Lecture 19. Monday, May 9, 2022

We have proved the following:

<u>Theorem</u> If X is Polish, then $(\mathcal{P}_2(X), W_2)$ is a metric space.

$$W_2(\mu,\nu) = \left[\min_{\gamma \in \mathcal{A}(\mu,\nu)} \int_{X\times X} d^2(x,y)\, d\gamma(x,y) \right]^{1/2} \quad \forall \gamma \in \mathcal{A}(\mu,\nu).$$

We now study other results about $(\mathcal{P}_2(X), W_2)$

<u>Theorem</u> If X is a Polish space, then $(\mathcal{P}_2(X), W_2)$ is also a Polish space.

We have proved the completeness of $(\mathcal{P}_2(X), W_2)$. We will prove that $(\mathcal{P}_2(X), W_2)$ is separable <u>today</u>.

<u>Theorem</u> (Convergence in $(\mathcal{P}_2(X), W_2)$) Let X be Polish and $\mu_n, \mu \in \mathcal{P}_2(X)$ $(n=1,2,\cdots)$.

(1) $\mu_n \xrightarrow{W_2} \mu \Longrightarrow \mu_n \to \mu$ narrowly and $\forall x_0 \in X$

$$\int_X d^2(x,x_0)\, d\mu_n(x) \longrightarrow \int_X d^2(x,x_0)\, d\mu(x). \qquad (*)$$

(2) $\mu_n \longrightarrow \mu$ narrowly and $(*)$ holds true for some $x_0 \Longrightarrow \mu_n \xrightarrow{W_2} \mu$.

We will prove this main result in next lecture.

<u>Corollary</u> If X is a compact metric space then $(\mathcal{P}_2(X), W_2) = (\mathcal{P}(X), W_2)$ is a compact metric space.

We have proved this, cf. Lecture 18.

<u>Theorem</u> If X is Polish, then $(\mathcal{P}_2(X), W_2)$ is also Polish.

The completeness of $(\mathcal{P}_2(X), W_2)$ has been proven. Here, we use the convergence theorem to prove the separability.

Proof Need only to prove that $(\mathcal{P}_2(X), W_2)$ is separable. Let D be a dense subset of X. Let Z be the collection of probability measures $\sum q_i \delta_{x_i}$ (finite sum), $q_i \in \mathbb{Q}$, $q_i \geq 0$, $\sum_i q_i = 1$, $x_i \in D$. Clearly, $Z \subseteq \mathcal{P}_2(X)$. Let $\overline{Z}$ be the closure of Z w.r.t. W_2. $\overline{Z} \subseteq \mathcal{P}_2(X)$ as $(\mathcal{P}_2(X), W_2)$ is complete. Since D is dense in X, $\overline{Z}$ contains all probability measures in $\mathcal{P}_2(X)$ of the form $\sum_i t_i \delta_{x_i}$ with a countable sum, and $t_i \in \mathbb{R}_+$, $\sum_i t_i = 1$, $x_i \in X$. In fact, we can approximate any finite sum $\sum_i t_i \delta_{x_i}$ with $t_i \geq 0$, $\sum_i t_i = 1$, $x_i \in X$ by measures in Z, using the characterization of convergence in $(\mathcal{P}_2(X), W_2)$ and the fact that $W_2(\delta_x, \delta_y) = d(x,y)$ $\forall x, y \in X$. Then, we can approximate countable sum $\sum t_i \delta_{x_i}$ by a sequence of finite sums.

Claim: If $\mu \in \mathcal{P}_2(X)$ and $\operatorname{supp}(\mu)$ is bounded, then $\mu \in \overline{Z}$. Proof of Claim. Let $\mu \in \mathcal{P}_2(X)$ and $\operatorname{supp}(\mu) \subseteq B(x_0, R)$ for some $x_0 \in X$ and $R > 0$. $\forall k \in \mathbb{N}$. Partition $B(x_0, R)$ into countably many disjoint Borel sets $A_{k,i}$ $(i = 1, 2, \cdots)$ of diameter $\leq \frac{1}{k}$. Set $\mu_k = \sum_i t_{k,i} \delta_{x_{k,i}}$ with $x_{k,i} \in A_{k,i}$ and $t_{k,i} = \mu(A_{k,i})$. Note that each $\mu_k \in Z$, and $\mu_k \in \mathcal{P}_2(X)$ as

$$\int_X d^2(x, x_0)\, d\mu_k(x) = \sum_i t_{k,i}\, d^2(x_{k,i}, x_0) \leq \sum_i t_{k,i} R^2 = R^2.$$

Now, for any $f \in BL(X)$,

$$\left| \int_{A_{k,i}} f\, d\mu - t_{k,i} \int_{A_{k,i}} f\, d\delta_{x_{k,i}} \right| = \left| \int_{A_{k,i}} f\, d\mu - \mu(A_{k,i}) f(x_{k,i}) \right|$$

$$= \left| \int_{A_{k,i}} [f - f(x_{k,i})] d\mu \right| \le \int_{A_{k,i}} \mathrm{Lip}(f)\, d(x, x_{k,i})\, d\mu(x)$$

$$\le \frac{1}{k} \mathrm{Lip}(f)\, \mu(A_{k,i}).$$

Thus.

$$\left| \int_X f\, d\mu - \int_X f\, d\mu_k \right| \le \sum_i \left| \int_{A_{k,i}} f\, d\mu - \int_{A_{k,i}} f\, d\mu_k \right|$$

$$= \sum_i \left| \int_{A_{k,i}} f\, d\mu - t_{k,i} \int_{A_{k,i}} f\, d\delta_{x_{k,i}} \right|$$

$$\le \sum_i \frac{1}{k} \mathrm{Lip}(f)\, \mu(A_{k,i}) = \frac{1}{k} \mathrm{Lip}(f) \to 0.$$

Hence $\mu_k \to \mu$ narrowly.

Replacing f by $d(\cdot, x_0)$ (which may not be Lipschitz) and noting that for $x \in A_{k,i}$

$$|d^2(x, x_0) - d^2(x_{k,i}, x_0)|$$

$$\le |d(x, x_0) - d(x_{k,i}, x_0)|\, [d(x, x_0) + d(x_{k,i}, x_0)]$$

$$\le d(x, x_{k,i}) \cdot 2R \le \frac{2}{k} R,$$

we get by the same argument

$$\int_X d^2(x, x_0)\, d\mu_k(x) \longrightarrow \int_X d^2(x, x_0)\, d\mu(x).$$

Thus, $\mu_k \xrightarrow{W_2} \mu$ by the convergence theorem, and $\mu \in \overline{Z}$.

It remains to show that any $\mu \in \mathcal{P}_2(X)$ can be W_2-approximated by those in $\mathcal{P}_2(X)$ of bounded support.

In fact, fix $\mu \in \mathcal{P}_2(X)$ and $x_0 \in \mathrm{Supp}(\mu)$. Define $\mu_k = \frac{1}{\mu(B(x_0, k))} \mu \llcorner B(x_0, k)$ $(k = 1, 2, \cdots)$. Then, each μ_k has a bounded support. Clearly, $\mu_k(X) = 1$, and

$$\int_X d^2(x, x_0)\, d\mu_k(x) = \frac{1}{\mu(B(x_0, k))} \int_{B(x_0, k)} d^2(x, x_0)\, d\mu(x) \le k^2 < \infty.$$

Hence, $\mu_k \in \mathcal{P}_2(X)$. Moreover, $B(x_0,k)\uparrow X$, so, $\mu(B(x_0,k)) \to \mu(X)$, and $\mathbb{1}_{B(x_0,k)} \uparrow \mathbb{1}_X$. Hence, by the monotone convergence theorem,

$$\int_X d^2(x,x_0)\,d\mu_k(x) = \frac{1}{\mu(B(x_0,k))}\int_X \mathbb{1}_{B(x_0,k)}\, d^2(x,x_0)\,d\mu(x)$$

$$\longrightarrow \int_X d^2(x,x_0)\,d\mu(x).$$

In addition, $\mu_k \to \mu$ narrowly, as $\forall f \in C_b(X)$

$$\int_X f\,d\mu_k = \frac{1}{\mu(B(x_0,k))}\int_X \mathbb{1}_{B(x_0,k)}\, f\,d\mu \to \int_X f\,d\mu$$

by the dominated convergence theorem. Thus, by the main convergence theorem, $\mu_k \xrightarrow{W_2} \mu$. QED