Lecture 24, Friday, 5/20/2022

<u>Theorem</u> (Kantorovich duality) Let X, Y be Polish, $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$, $c: X \times Y \to [0, \infty]$ lower semi-continuous. Then

$$\inf_{\gamma \in \mathcal{A}(\mu,\nu)} E_K[\gamma] = \sup_{(\varphi,\psi) \in \Phi_c(\mu,\nu)} J[\varphi,\psi].$$

Recall $\mathcal{A}(\mu,\nu) = \{\gamma \in \mathcal{P}(X \times Y) : \pi^X_\# \gamma = \mu,\ \pi^Y_\# \gamma = \nu\}$.

$\Phi_c(\mu,\nu) = \{(\varphi,\psi) \in L^1(\mu) \times L^1(\nu) : \varphi(x) + \psi(y) \le c(x,y),$
$\mu\text{-a.e. } x \in X,\ \nu\text{-a.e. } y \in Y\}$.

$$E_K[\gamma] = \int_{X \times Y} c(x,y)\, d\gamma(x,y).$$

$$J[\varphi,\psi] = \int_X \varphi\, d\mu + \int_Y \psi\, d\nu.$$

<u>Lemma</u> (Weak duality) $\forall \gamma \in \mathcal{A}(\mu,\nu)\ \forall (\varphi,\psi) \in \Phi_c(\mu,\nu)$:

$J[\varphi,\psi] \le E_K[\gamma]$, and $\sup_{\Phi_c(\mu,\nu)} J \le \inf_{\mathcal{A}(\mu,\nu)} E_K$.

<u>Proof</u>: See last lecture. <u>QED</u>

<u>Definition</u> (Legendre-Fenchel transform) Let E be a (real) normed vector space and E^* its dual space. Let $f: E \to \mathbb{R} \cup \{\infty\}$ be a convex function. The Legendre-Fenchel transform of f is $f^*: E^* \to \mathbb{R}$,

$$f^*(x^*) = \sup_{x \in E} [\langle x^*, x\rangle - f(x)],$$

where $\langle x^*, x\rangle = x^*(x)$.

<u>Theorem</u> (Fenchel-Rockafellar duality) Let E be a normed vector space and E^* its dual. Let $f, g: E \to \mathbb{R} \cup \{+\infty\}$ be convex. Assume $\exists x_0 \in E$

such that $f(x_0)<\infty$, $g(x_0)<\infty$, and f is continuous at x_0. Then

$$\inf_E (f+g) = \max_{x^*\in E^*}\left[-f^*(-x^*)-g^*(x^*)\right].$$

Proof: See last lecture. QED

Theorem (Duality for compact spaces X and Y). Assume X and Y are compact and $c: X\times Y\to[0,\infty)$ is continuous. The The conclusion of the theorem is true.

Proof Define $f, g: C_b(X\times Y)\longrightarrow \mathbb{R}\cup\{+\infty\}$ by

$$f(u)=\begin{cases} 0 & \text{if } u(x,y)\geq -c(x,y) \quad \forall (x,y)\in X\times Y, \\ +\infty & \text{otherwise.}\end{cases}$$

$$g(u)=\begin{cases} \int_X \varphi\, d\mu + \int_Y \psi\, d\nu & \text{if } u(x,y)=\varphi(x)+\psi(y) \ \forall (x,y)\in X\times Y \\ & \text{for some } \varphi\in C_b(X),\ \psi\in C_b(Y). \\ +\infty & \text{otherwise.}\end{cases}$$

Note that g is well-defined: $u(x,y)=\varphi(x)+\psi(y)=\tilde\varphi(x)+\tilde\psi(y)$
$\Rightarrow \varphi(x)-\tilde\varphi(x)=\tilde\psi(y)-\psi(y)=a\in\mathbb{R}$ for all $x\in X$, $y\in Y$.
$\Rightarrow \varphi(x)=\tilde\varphi(x)+a$ and $\psi(y)=\tilde\psi(y)-a$ $\forall x, y$. Thus

$$\int_X \varphi\, d\mu+\int_Y \psi\, d\nu=\int_X(\tilde\varphi+a)\,d\mu+\int_Y(\tilde\psi-a)\,d\nu=\int_X\tilde\varphi\, d\mu+\int_Y\tilde\psi\, d\nu.$$

Since $f=0$ in $\mathrm{Dom}(f)=\{u\in C_b(X\times Y): f(u)<\infty\}$, f is convex. Similarly, g is linear in $\mathrm{Dom}(g)$, g is convex. Let $u_0(x,y)=1$ $\forall (x,y)$. So, $u_0\in C_b(X\times Y)$, and $f(u_0)<\infty$, $g(u_0)<\infty$, and f is continuous at u_0.

Since X and Y are compact, by Riesz's Thm, $[C_b(X\times Y)]^*=\mathcal{M}(X\times Y)$ (finite signed measures with the norm being the total variation). Now, applying the Fenchel-Rockafellar duality with

$E = C_b(X\times Y)$, $E^* = \mathcal{M}(X\times Y)$, and with u_0 replacing x_0 in the theorem, we have

$$\inf_{u\in C_b(X\times Y)} [f(u)+g(u)] = \max_{\gamma\in\mathcal{M}(X\times Y)} [-f^*(-\gamma)-g^*(\gamma)]. \quad (*)$$

The l.h.s. of $(*)$ is, after replacing (φ,ψ) by $(-\varphi,-\psi)$,

$$\inf\left\{\int_X \varphi\,d\mu + \int_Y \psi\,d\nu : \varphi(x)+\psi(y) \geq -c(x,y)\ \forall (x,y)\in X\times Y,\ \varphi\in C_b(X),\ \psi\in C_b(Y)\right\}$$

$$= -\sup\{ J[\varphi,\psi] : (\varphi,\psi)\in \Phi_c(\mu,\nu)\cap(C_b(X)\times C_b(Y))\}. \quad (**)$$

To calculate ~~the~~ r.h.s. of $(*)$, let us fix $\gamma\in\mathcal{M}(X\times Y)$.

$$f^*(-\gamma) = \sup\left\{-\int_{X\times Y} u\,d\gamma : u\in C_b(X\times Y),\ u\geq -c\right\}$$

$$= \sup\left\{\int_{X\times Y} u\,d\gamma : u\in C_b(X\times Y),\ u\leq c\right\}.$$

If $\gamma\notin\mathcal{M}_+(X\times Y)$ then $\exists\, v\in C_b(X\times Y)$, $v\geq 0$ s.t. $\int_{X\times Y} v\,d\gamma<0$.

Let $u_\lambda = -\lambda v \in C_b(X\times Y)$ with $\lambda>0$. Then, $u_\lambda\leq 0\leq c$. Moreover, $\int_{X\times Y} u_\lambda\,d\gamma\to\infty$. Hence, $f^*(-\gamma)=\infty$.

If $\gamma\in\mathcal{M}_+(X\times Y)$ then $\int_{X\times Y} u\,d\gamma \leq \int_{X\times Y} c\,d\gamma$ if $u\leq c$, hence

Since c is continuous, and $X\times Y$ is compact, $c\in C_b(X\times Y)$.

Thus, $f^*(-\gamma) = \int_{X\times Y} c\,d\gamma$. Hence

$$f^*(-\gamma) = \begin{cases} \int_{X\times Y} c\,d\gamma & \text{if } \gamma\in\mathcal{M}_+(X\times Y), \\ +\infty & \text{if } \gamma\notin\mathcal{M}_+(X\times Y). \end{cases} \quad (***)$$

Now, consider $\gamma\in\mathcal{M}(X\times X)$ again. Since $g(u)=+\infty$ if $u\notin \mathrm{Dom}$, we have

$$g^*(\gamma) = \sup_{u\in C_b(X\times Y)} [\langle\gamma,u\rangle - g(u)]$$

$$= \sup_{u \in \mathrm{Dom}(g)} [\langle \gamma, u \rangle - g(u)].$$

Assume $\gamma \in \mathcal{A}(\mu, \nu)$. If $u \in \mathrm{Dom}(g)$, then $u(x,y) = \varphi(x) + \psi(y)$ $\forall x \in X$, $\forall y \in Y$, for some $\varphi \in C_b(X)$, $\psi \in C_b(Y)$, and $g(u) = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu$. Thus,

$$\langle \gamma, u \rangle - g(u) = \int_{X \times Y} u \, d\gamma - \int_X \varphi \, d\mu - \int_Y \psi \, d\nu$$

$$= \int_{X \times Y} u \, d\gamma - \int_{X \times Y} \varphi(x) \, d\gamma(x,y) - \int_{X \times Y} \psi(y) \, d\gamma(x,y)$$

$$= 0.$$

Hence, $\gamma \in \mathcal{A}(\mu, \nu) \Rightarrow g^*(\gamma) = 0$. Assume $\gamma \notin \mathcal{A}(\mu, \nu)$. Then $\pi^X_\# \gamma \neq \mu$, and $\exists \varphi \in C_b(X)$ s.t. $\int_X \varphi \, d\mu \neq \int_{X \times Y} \varphi(x) \, d\gamma(x,y)$, set $u(x,y) = \varphi(x)$, or $\pi^Y_\# \gamma \neq \nu$, and $\exists \psi \in C_b(Y)$ s.t. $\int_Y \psi \, d\nu \neq \int_{X \times Y} \psi(y) \, d\gamma(x,y)$, set $u(x,y) = \psi(y)$. In any case, $\exists \hat{u} \in \mathrm{Dom}(g)$ s.t. $\langle \gamma, \hat{u} \rangle \neq g(\hat{u})$. Then, since for any $\lambda \in \mathbb{R}$, $\lambda \hat{u} \in \mathrm{Dom}(g)$ and $\langle \gamma, \lambda \hat{u} \rangle - g(\lambda \hat{u}) = \lambda [\langle \gamma, u \rangle - g(u)]$. We have

$$g^*(\gamma) \geq \sup_{\lambda \in \mathbb{R}} [\langle \gamma, \lambda \hat{u} \rangle - g(\lambda \hat{u})] = +\infty.$$

Thus,

$$g^*(\gamma) = \begin{cases} 0 & \text{if } \gamma \in \mathcal{A}(\mu, \nu), \\ +\infty & \text{otherwise}. \end{cases} \qquad (****)$$

Since $\mathcal{A}(\mu, \nu) \subseteq \mathcal{M}_+(\mu, \nu)$, it now follows from (*)—(****) that

$$-\sup \{ J[\varphi, \psi] : (\varphi, \psi) \in \Phi_c(\mu, \nu) \cap (C_b(X) \times C_b(Y)) \}$$

$$= \max_{\gamma \in \mathcal{A}(\mu, \nu)} \left(- \int_{X \times Y} c \, d\gamma \right) = - \min_{\gamma \in \mathcal{A}(\mu, \nu)} \int_{X \times Y} c \, d\gamma.$$

which is
$$\sup_{\Phi_c(\mu,\nu)\cap(C_b(X)\times C_b(Y))} J[\cdot] = \min_{\mathcal{A}(\mu,\nu)} E_c[\cdot]. \quad (\star)$$

Finally, for any $(\varphi,\psi)\in L^1(\mu)\times L^1(\nu)$ there exist $(\varphi_n,\psi_n)\in \Phi_C(\mu,\nu)\cap(C_b(X)\times C_b(Y))$ s.t. (see below.)
$$\int_X \varphi_n\,d\mu \to \int_X \varphi\,d\mu \quad\text{and}\quad \int_Y \psi_n\,d\mu \to \int_Y \psi\,d\mu.$$
Hence, we can replace C_b by L^1 in $(\star)$. QED

Proposition Let (X,d) be a Polish space, $\mu\in\mathcal{P}(X)$, and $\varphi\in L^1(\mu)$. There exist $\varphi_n\in C_b(X)$ $(n=1,2,\cdots)$ s.t.
$$\int_X \varphi_n\,d\mu \to \int_X \varphi\,d\mu.$$

Proof Since μ is finite, and $L^1(\mu)$-functions can be approximated by simple functions, we need only to consider $\varphi = \mathbb{1}_A$ for $A\in\mathcal{B}(X)$. By the regularity of μ, $\forall\varepsilon>0$, $\exists\, U\subseteq X$, $K\subseteq X$, s.t. U is open, K is compact, $K\subseteq A\subseteq U$, $\mu(U\setminus K)<\varepsilon$. Define $\varphi_\varepsilon(x) = \dfrac{d(x,U^c)}{d(x,K)+d(x,U^c)}$, $x\in X$. Then $\varphi_\varepsilon\in C_b(X)$. In fact $0\le\varphi_\varepsilon\le 1$. $\varphi_\varepsilon = 1$ on K. $\varphi_\varepsilon=0$ on U^c. Moreover,
$$\left|\int_X \mathbb{1}_A\,d\mu - \int_X \varphi_\varepsilon\,d\mu\right| = \left|\int_{A\setminus K} d\mu - \int_{U\setminus K}\varphi_\varepsilon\,d\mu\right|$$
$$\le \mu(A\setminus K)+\mu(U\setminus K) \le 2\,\mu(U\setminus K) = 2\varepsilon.$$ QED