Lecture 28, Wed., 6/1/2022

<u>Review</u> Given $c: X\times Y \to \mathbb{R}$. $\Gamma \subseteq X\times Y$ is c-cyclically monotone, if

$$\sum_{i=1}^{N} c(x_i, y_i) < \sum_{i=1}^{N} c(x_i, y_{\sigma(i)})$$

for any $N\in\mathbb{N}$, any $(x_i, y_i)\in\Gamma$ $(i=1,\dots,N)$, any permutation σ of $\{1,\dots,N\}$.

<u>Theorem</u> Let X and Y be Polish spaces, $\mu\in\mathcal{P}(X)$, $\nu\in\mathcal{P}(Y)$, and $c: X\times Y\to[0,\infty)$ continuous. Let $\gamma = \arg\min_{\mathcal{A}(\mu,\nu)} E_K$ with $E_K[\gamma]=\int_{X\times Y} c\, d\gamma$, $\min_{\mathcal{A}(\mu,\nu)} E_K<\infty$. Then, $\operatorname{supp}\gamma\subseteq X\times Y$ is c-cyclically monotone.

<u>Today</u> ⊙ Definition/properties of c-concave functions, and subdifferential.

⊙ Theorem: a generalized Rockafellar Theorem

<u>Theorem</u> (A generalized Rockafellar theorem). Let X and Y be two nonempty sets and $c: X\times Y\to\mathbb{R}$ a function. Let $\Gamma\subseteq X\times Y$ be c-cyclically monotone. Then, there exists a c-concave function $\varphi: X\to[-\infty,\infty)$, $\phi\not\equiv-\infty$, such that $\Gamma\subseteq \operatorname{Graph}(\partial^c\phi)$.

<u>c-Concave functions</u>

<u>Definition</u> Let X, Y be nonempty sets and $c: X\times Y\to\mathbb{R}$ a function. For any $\varphi: X\to[-\infty,\infty)$ and $\psi: Y\to[-\infty,\infty)$, we define

$$\varphi^c(y) = \inf_{x\in X}[c(x,y)-\varphi(x)] \quad \forall y\in Y.$$

$$\psi^c(x) = \inf_{y\in Y}[c(x,y)-\psi(y)] \quad \forall x\in X.$$

Call φ^c, ψ^c <u>c-concave functions</u> (or <u>c-conjugate</u>

functions) of φ, ψ, respectively. Call $(\varphi^{cc}, \varphi^c)$ a c-conjugate pair (if $\varphi^c \neq +\infty$).

Remark Sometimes we include $-\infty$, or $+\infty$. The rule is that we cannot have $\infty - \infty$, as that is not defined.

Definition (c-Concavity) A function $\varphi : X \to [-\infty, \infty)$ is c-concave if it is the infimum of a family of c-affine functions $c(\cdot, y) + \alpha$, i.e.,

$$\varphi(x) = \inf_{y \in B} \{ c(x, y) + \alpha_y \},$$

for some $B \subseteq Y$ and $\alpha_y \in \mathbb{R}$ $(y \in B)$. A function $\psi : Y \to [-\infty, \infty)$ is c-concave if it is the infimum of a family of c-affine functions $C(x, \cdot) + \beta$.

Example $X = Y = \mathbb{R}^d$. $c(x,y) = \frac{1}{2}|x-y|^2 = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - x \cdot y$.
$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$. $\gamma \in \mathcal{A}(\mu, \nu)$

$$E_K[\gamma] = \int_{X \times Y} c(x,y) \, d\gamma(x,y)$$

$$= \int_{X \times Y} \frac{1}{2}|x|^2 d\gamma(x,y) + \int_{X \times Y} \frac{1}{2}|y|^2 d\gamma(x,y) - \int_{X \times Y} x \cdot y \, d\gamma(x,y)$$

$$= \int_{\mathbb{R}^d} \frac{1}{2}|x|^2 d\mu(x) + \int_{\mathbb{R}^d} \frac{1}{2}|y|^2 d\nu(y) - \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot y \, d\gamma(x,y)$$

If $\hat{c}(x,y) = -x \cdot y$, then for $\varphi : \mathbb{R}^d \to [-\infty, \infty)$ we have

$$\varphi^{\hat{c}}(y) = \inf_{x \in \mathbb{R}^d} \{ -x \cdot y - \varphi(x) \} = - \sup_{x \in \mathbb{R}^d} \{ x \cdot y - (-\varphi(x)) \}.$$

$$= -(-\varphi)^*(y).$$ $(-\varphi)^*$ is the Legendre transform of $-\varphi$.

Definition (C-Subdifferential). Let X and Y be two nonempty sets and $c: X\times Y \to \mathbb{R}$. Let $\phi: X \to [-\infty, \infty)$ and $x\in X$ with $\varphi(x)\in\mathbb{R}$. Define the c-subdifferential of φ at x as

$$\partial^c\varphi(x) = \{y\in Y : c(z,y)-\phi(z) \text{ is minimal at } z=x\}.$$

Define the graph of $\partial^c\varphi$ to be

$$\mathrm{Graph}(\partial^c\varphi) = \bigcup_{x\in X}\{x\}\times\partial^c\varphi(x).$$

Remark ⊙ The minimal condition is

$$c(x,y)-\phi(x) \le c(z,y)-\phi(z) \quad \forall z\in Y,$$

i.e.,
$$\phi(z) \le \phi(x) - c(x,y) + c(z,y) \quad \forall z\in Y.$$

⊙ One can show that

$$y\in\partial^c\phi(x) \Leftrightarrow \phi(x)+\phi^c(y) = c(x,y) \Leftrightarrow x\in\partial^c\phi(y).$$

⊙ A subset $\Gamma\subseteq X\times Y$ satisfies $\Gamma\subseteq \mathrm{Graph}(\partial^c\varphi)$ if and only if

$$\phi(x) \le c(x,y_*) - c(x_*,y_*) + \varphi(x_*) \quad \forall (x_*,y_*)\in\Gamma \ \forall x\in X.$$

Theorem (Generalized Rockafellar Theorem). Let X and Y be nonempty sets and $c: X\times Y\to\mathbb{R}$. If $\Gamma\subseteq X\times Y$ is c-cyclically monotone, then there exists a c-concave function $\varphi: X\to[-\infty,\infty)$, $\phi\not\equiv-\infty$, such that $\Gamma\subseteq\mathrm{Graph}(\partial^c\varphi)$.

Proof Fix $(x_0,y_0)\in\Gamma$. Construct φ s.t. $\varphi(x_0)=0$. Note that $\Gamma\subseteq\mathrm{Graph}(\partial^c\varphi)$ is equivalent to

$\phi(x) \le c(x, y_*) - c(x_*, y_*) + \varphi(x_*) \quad \forall (x_*, y_*) \in \Gamma \ \forall x \in X.$

Now, choose $(x_1, y_1) \in \Gamma$ and apply this condition to (x_1, y_1) first and (x_0, y_0):

$$\varphi(x) \le c(x, y_1) - c(x_1, y_1) + \varphi(x_1)$$
$$\le c(x, y_1) - c(x_1, y_1) + c(x_1, y_0) - c(x_0, y_0).$$

Continuing this process, we get

$$\phi(x) \le \inf \{ c(x, y_N) - c(x_N, y_N) + c(x_N, y_{N-1})$$
$$- c(x_{N-1}, y_{N-1}) + c(x_{N-1}, y_{N-2}) - c(x_{N-2}, y_{N-2})$$
$$+ \cdots + c(x_1, y_0) - c(x_0, y_0) \}$$
$$= \inf \sum_{i=1}^{N+1} [c(x_i, y_{i-1}) - c(x_{i-1}, y_{i-1})], \quad (*)$$

where $x_{N+1} = x$, and the infimum is taken over all $N \ge 1$ and $(x_1, y_1), \cdots, (x_N, y_N) \in \Gamma$.

Define $\varphi : X \to [-\infty, \infty)$ by $(*)$.

We check:

(1) $\phi : X \to [-\infty, \infty)$. True since $c : X \times Y \to \mathbb{R}$.

(2) ϕ is c-concave, as it is the infimum of c-affine functions $c(\cdot, y_N) + \alpha_{y_N}$.

(3) $\Gamma \subseteq \mathrm{Graph}(\partial^c \varphi)$, i.e.,

$\phi(x) \le c(x, y_*) - c(x_*, y_*) + \varphi(x_*) \quad \forall (x_*, y_*) \in \Gamma \ \forall x \in X.$

From $(*)$, keeping (x_N, y_N) fixed, then

$$\phi(x) = \inf_{(x_N, y_N) \in \Gamma} \left[c(x, y_N) - c(x_N, y_N) + \varphi(x_N) \right].$$

Thus, (*) is true.

(4) $\varphi(x_0) = 0$. First,

$$\varphi(x_0) \le c(x_0, y_0) - c(x_0, y_0) = 0.$$

Now, since Γ is c-cyclically monotone, with particularly the cyclic permutation $x_i \mapsto x_{i+1}$ $(1 \le i \le N)$ and $x_N \mapsto x_0$, we get

$$\sum_{i=1}^{N+1} \left[c(x_i, y_{i-1}) - c(x_{i-1}, y_{i-1}) \right] \ge 0.$$

Hence $\varphi(x) \ge 0$. Particularly, $\varphi(x_0) \ge 0$. So, $\varphi(x_0) = 0$.

QED

Where are we now?

$X = Y = \mathbb{R}^d$, $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$. $c(x,y) = \frac{1}{2}|x-y|^2$

$$\max_{\mathcal{A}(\mu,\nu)} E_K[\cdot] = \int_{\mathbb{R}^d} \frac{1}{2}|x|^2 d\mu(x) + \int_{\mathbb{R}^d} \frac{1}{2}|y|^2 d\nu(y) - \inf_{\mathcal{A}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot y \, d\gamma(x,y)$$

If $\gamma = \operatorname{argmin}_{\mathcal{A}(\mu,\nu)} E_K[\cdot]$. Then

- $\operatorname{supp} \gamma \subseteq \mathbb{R}^d \times \mathbb{R}^d$ is c-cyclically monotone.
- $\operatorname{supp} \gamma \subseteq \operatorname{Graph}(\partial\varphi)$ for some convex function $\varphi: \mathbb{R}^d \to [-\infty, \infty)$.

Next: $T = \nabla\varphi$. $\gamma = (Id \times T)_{\#}\mu$, and uniqueness.