Math 231B: Partial Differential Equations
Winter quarter 2015

Exercises on the Calculus of Variations

1. Let a, $b \in \mathbb{R}$ and $0 < \lambda < 1$. Define $u(x) = \lambda a + (1-\lambda)b$ for $x \in (0,1)$. For each integer $k \geq 1$, define $u_k \in L^2(0,1)$ by
$$u_k(x) = \begin{cases} a & \text{if } j/k \leq x < (j+\lambda)/k \\ b & \text{if } (j+\lambda)/k \leq x < (j+1)/k \end{cases} \qquad (j = 0, \ldots, k-1).$$
Prove that $\{u_k\}_{k=1}^{\infty}$ converges weakly to u in $L^2(0,1)$.

2. Let X be a Banach space. Prove that any convex and continuous functional $I : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous.

3. Define $I : W^{1,4}(0,1) \to \mathbb{R}$ by
$$I[u] = \int_0^1 \left[(u'^2 - 1)^2 + u^2\right] dx.$$
Prove that $\inf_{u \in W^{1,4}(0,1)} I[u] = 0$ but there exists no $u \in W^{1,4}(0,1)$ such that $I[u] = 0$.

4. Let $U \subset \mathbb{R}^n$ be a bounded open set. Define
$$I[u] = \int_\Omega \frac{1}{2}\left[|\nabla u|^2 - \log\left(1+u^2\right)\right] dx \qquad \forall u \in H_0^1(\Omega).$$
Prove there exists a minimizer of $I : H_0^1(U) \to \mathbb{R}$ over $H_0^1(U)$. Derive the corresponding Euler–Lagrange equation.

5. Let U be a bounded open subset of R^n with a smooth boundary ∂U. Let $\varepsilon \in L^\infty(U)$ be such that $\varepsilon(x) \geq \varepsilon_{\min}$ for all $x \in U$, where $\varepsilon_{\min}$ is a positive number. Let $g \in H^1(U) \cap C(\overline{U})$ and define $H_g^1(U) = \{u \in H^1(U) : u = g \text{ on } \partial U\}$. Let $V \in C^2(\mathbb{R})$ be a strictly convex function such that $V(0) = 0 < V(s)$ for any $s \neq 0$ and $V(\pm\infty) = \infty$. Define
$$I[u] = \int_U \left[\frac{\varepsilon}{2}|\nabla u|^2 + V(u)\right] dx \qquad \forall u \in H_g^1(U).$$
Prove that there exists a unique $u \in H_g^1(U)$ such that $I[u] = \min_{v \in H_g^1(U)} I[v]$. Moreover, $u \in L^\infty(U)$, and u is the unique weak solution to the boundary-value problem
$$\begin{cases} -\nabla \cdot \varepsilon \nabla u + V'(u) = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

6. Let U be a bounded open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. Let X be the subset of $L^2(U;\mathbb{R}^n)$ consisting of all $\xi \in L^2(U;\mathbb{R}^n)$ such that $\operatorname{div} \xi = f$ in the weak sense, i.e.,
$$-\int_U \xi \cdot \nabla \phi \, dx = \int_U f \, \phi \, dx \quad \forall \phi \in C_c^1(U).$$
Prove the dual variational principle that
$$\min_{v \in H_0^1(U)} \int_U \left(\frac{1}{2}|\nabla v|^2 - fv\right) dx = \max_{\xi \in X} -\frac{1}{2}\int_U |\xi|^2 dx.$$