Math 231B: Partial Differential Equations  
Winter quarter 2015

Exercises on the Calculus of Variations

1. Let $a, b \in \mathbb{R}$ and $0 < \lambda < 1$. Define $u(x) = \lambda a + (1 - \lambda)b$ for $x \in (0, 1)$. For each integer $k \geq 1$, define $u_k \in L^2(0, 1)$ by

$$u_k(x) = \begin{cases} a & \text{if } j/k \leq x < (j + \lambda)/k \\ b & \text{if } (j + \lambda)/k \leq x < (j + 1)/k \end{cases} \quad (j = 0, \ldots, k - 1).$$

Prove that $\{u_k\}_{k=1}^\infty$ converges weakly to $u$ in $L^2(0, 1)$.

2. Let $X$ be a Banach space. Prove that any convex and continuous functional $I : X \to \mathbb{R}$ is sequentially weakly lower semicontinuous.

3. Define $I : W^{1,4}(0, 1) \to \mathbb{R}$ by

$$I[u] = \int_0^1 [(u'^2 - 1)^2 + u^2] \, dx.$$  

Prove that $\inf_{u \in W^{1,4}(0, 1)} I[u] = 0$ but there exists no $u \in W^{1,4}(0, 1)$ such that $I[u] = 0$.

4. Let $U \subset \mathbb{R}^n$ be a bounded open set. Define

$$I[u] = \int_\Omega \frac{1}{2} \left[ |\nabla u|^2 - \log (1 + u^2) \right] \, dx \quad \forall u \in H^1_0(\Omega).$$

Prove there exists a minimizer of $I : H^1_0(U) \to \mathbb{R}$ over $H^1_0(U)$. Derive the corresponding Euler–Lagrange equation.

5. Let $U$ be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial U$. Let $\varepsilon \in L^\infty(U)$ be such that $\varepsilon(x) \geq \varepsilon_{\text{min}}$ for all $x \in U$, where $\varepsilon_{\text{min}}$ is a positive number. Let $g \in H^1(U) \cap C(\overline{U})$ and define $H^1_g(U) = \{ u \in H^1(U) : u = g$ on $\partial U \}$. Let $V \in C^2(\mathbb{R})$ be a strictly convex function such that $V(0) = 0 < V(s)$ for any $s \neq 0$ and $V(\pm \infty) = \infty$. Define

$$I[u] = \int_U \left[ \frac{\varepsilon}{2} |\nabla u|^2 + V(u) \right] \, dx \quad \forall u \in H^1_g(U).$$

Prove that there exists a unique $u \in H^1_g(U)$ such that $I[u] = \min_{v \in H^1_g(U)} I[v]$. Moreover, $u \in L^\infty(U)$, and $u$ is the unique weak solution to the boundary-value problem

$$\begin{cases} -\nabla \cdot \varepsilon \nabla u + V'(u) = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

6. Let $U$ be a bounded open subset of $\mathbb{R}^n$ and $f \in L^2(U)$. Let $X$ be the subset of $L^2(U; \mathbb{R}^n)$ consisting of all $\xi \in L^2(U; \mathbb{R}^n)$ such that $\text{div} \xi = f$ in the weak sense, i.e.,

$$-\int_U \xi \cdot \nabla \phi \, dx = \int_U f \phi \, dx \quad \forall \phi \in C^1_c(U).$$

Prove the dual variational principle that

$$\min_{v \in H^1_0(U)} \int_U \left( \frac{1}{2} |\nabla v|^2 - fv \right) \, dx = \max_{\xi \in X} \left( -\frac{1}{2} \int_U |\xi|^2 \, dx \right).$$