Math 240A: Real Analysis, Fall 2012
Additional Exercise Problems

1. Let $(X, \rho)$ be a metric space. Define \(d : X \times X \to \mathbb{R}\) by
   \[
d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} \quad \forall x, y \in X.
   \]
   Prove that $(X, d)$ is also a metric space.

2. Let $(X, \rho)$ be a metric space and $E$ a closed subset of $X$. Define \(d : X \to \mathbb{R}\) by
   \[
d(x) = \inf_{y \in E} \rho(x, y).
   \]
   For each integer $n \geq 1$ define \(f_n : X \to [0, \infty)\) by
   \[
f_n(x) = \frac{1}{1 + nd(x)} \quad \forall x \in X.
   \]
   Prove that $0 \leq f_n \leq 1$ for each $n$, \(\{f_n\}_{n=1}^\infty\) is decreasing, and \(\lim_{n \to \infty} f_n(x) = \chi_E(x)\) for all $x \in X$.

3. Let $(X, \rho)$ be a metric space. If a Cauchy sequence \(\{x_n\}\) in $X$ has a subsequence that converges to some $x \in X$. Then \(\{x_n\}\) itself converges to $x$.

4. Prove that any discrete metric space is complete.

5. Let $X$ and $Y$ be two nonempty sets and $f : X \to Y$ a mapping. For any $T \subseteq Y$ define $f^{-1}(T) = \{x \in X : f(x) \in T\}$. Let $\mathcal{M} = \{f^{-1}(E) : E \subseteq Y\}$. Show that $\mathcal{M}$ is a $\sigma$-algebra on $X$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu$ finite. Let $A_n \in \mathcal{M}$ \((n = 1, 2, \ldots)\) be such that $\sum_{n=1}^\infty \mu(A_n) < \infty$. Prove that
   \[
   \mu\left( \bigcap_{m=1}^\infty \bigcup_{k=m}^\infty A_k \right) = 0.
   \]

7. Let $\mathcal{B}_\mathbb{R}$ and $\mathcal{L}$ denote the Borel and Lebesgue $\sigma$-algebras on $\mathbb{R}$, respectively. Why \(\text{card } (\mathcal{L}) > \text{card } (\mathcal{B}_\mathbb{R})\)?

8. Let $a, b \in \mathbb{R}$ with $a < b$. Let
   \[
f(x) = \begin{cases} 
   \frac{1}{b-a} & \text{if } a < x < b, \\
   0 & \text{if } x \leq a \text{ or } x \geq b.
   \end{cases}
   \]
   Define $F : \mathbb{R} \to \mathbb{R}$ by
   \[
   F(x) = \int_{-\infty}^x f(t) \, dt.
   \]
(1) Show that the Lebesgue–Stieltjes measure \( \mu_F \) associated to \( F \) is a probability measure on \((\mathbb{R}, \mathcal{B}_\mathbb{R})\).
(2) Show that \( \mu_F(E) = 0 \) if \( E \in \mathcal{B}_\mathbb{R} \) and \( E \cap (a, b) = \emptyset \).
(3) Let \( G = (c, d) \) with \( c, d \in \mathbb{R} \) such that \( a < c < d < b \). What is \( \mu_F(G) \)?

9. Let \( X \) be a non-empty set and \( \mathcal{P}(X) \) its power set. Let \( \delta \) be the Dirac delta measure on \((X, \mathcal{P}(X))\) concentrated on a point \( y \in X \).
   (1) For any \( E \subseteq X \), calculate \( \int_X \chi_E \, d\mu \).
   (2) Let \( f : X \to [0, \infty) \) be a function. Show that it is measurable and calculate \( \int_X f \, d\mu \).

10. Let \( X \) be a non-empty set and \( \mathcal{P}(X) \) its power set. Let \( \mu \) be the the counting measure on \((X, \mathcal{P}(X))\). Given \( f : X \to [0, \infty) \). Find conditions under which \( \int_X f \, d\mu < \infty \).

11. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \( E \in \mathcal{M} \) be such that \( \mu(E) \) and \( \mu(E^c) \) are both positive and finite. Define \( f_n = \chi_E \) if \( n \) is odd and \( f_n = 1 - \chi_E \) if \( n \) is even. Show that

\[
\int_X \left( \liminf_{n \to \infty} f_n \right) \, d\mu < \liminf_{n \to \infty} \int_X f_n \, d\mu,
\]

i.e., the strict inequality in Fatou’s lemma can occur.

12. Exercise 7 on page 27.
