Math 240A: Real Analysis, Fall 2019
Midterm Exam
Solution
B. Li, November 3, 2019

Name ___________________________ ID number ___________________________

Note: This is a close-book and close-note exam. There are 5 problems of total 100 points. To get credit, you must show your work. Partial credit will be given to partial answers.

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1. (20 points) Determine if each of the following statements is true or false. If it is true, prove it. If it is false, give a counter example.

(1) Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if \( \{ x \in X : f(x) > r \} \) is measurable for any rational number $r$.

Solution.
(1) True. Let $a \in \mathbb{R}$. Then there exist rational numbers $r_k (k = 1, 2, \ldots)$ such that they decrease and converge to $a$. Then we have \( \{ f > a \} = \bigcup_{k=1}^{\infty} \{ f > r_k \} \). But each \( \{ f > r_k \} \in \mathcal{M} \). Hence \( \{ f > a \} \in \mathcal{M} \). Thus, $f$ is measurable.

(2) False. Example. $E = \mathbb{N}$. First, for any $x \in \mathbb{R}$, we have

\[
m(\{x\}) = m \left( \bigcap_{n=1}^{\infty} (x - 1/n, x + 1/n) \right) = \lim_{n \to \infty} m((x - 1/n, x + 1/n)) = \lim_{n \to \infty} \frac{2}{n} = 0.
\]

Hence

\[
m(\mathbb{N}) = \sum_{n=1}^{\infty} m(\{n\}) = 0.
\]
2. (20 points) Let $\mu$ be the Lebesgue–Stieltjes measure associated to the following increasing and right-continuous function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \begin{cases} 
0 & \text{if } x < 0, \\
x + 1 & \text{if } 0 \leq x < 1, \\
x^2 + 2 & \text{if } 1 \leq x < \infty.
\end{cases}$$

Calculate $\mu(\{0\})$, $\mu((1, 2])$, and $\mu([3, \infty))$.

Solution.
(1) $\mu(\{0\}) = \lim_{n \to \infty} \mu((-1/n, 1/n)) = \lim_{n \to \infty} [F(1/n) - F(-1/n)] = \lim_{n \to \infty} (1 + 1/n - 0) = 1$.
(2) $\mu((1, 2]) = F(2) - F(1) = (2^2 + 2) - (1^2 + 2) = 3$.
(3) $\mu([3, \infty)) \geq \mu((3, \infty)) \geq \mu((3, n]) = F(n) - F(3) \to \infty$ as $n \to \infty$. Hence $\mu([3, \infty)) = \infty$. 


3. (20 points) Calculate the following limit with justification: (Note that $|\sin u| \leq |u|$ for any $u \in \mathbb{R}$.)

$$\lim_{n \to \infty} \int_0^n \frac{n}{x(1+x^2)} \sin \left(\frac{x}{n}\right) dx.$$ 

**Solution.** Let $f_n(x) = \chi_{(0,n)}(x)\frac{n}{x(1+x^2)} \sin \left(\frac{x}{n}\right)$ for $0 < x < \infty$. Each $f_n$ is Lebesgue measurable as it is the product of two Lebesgue measurable functions, one a simple function and the other continuous function.

Note that $|f_n(x)| \leq 1/(1 + x^2)$ for all $x > 0$ since $|n \sin(x/n)/x| = |\sin(x/n)/(x/n)| \leq 1$ for all $n$ and all $x > 0$. Moreover, $1/(1 + x^2)$ is integrable in $(0, \infty)$. In addition, for any $x > 0$,

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \chi_{(0,n)}(x) \lim_{n \to \infty} \frac{\sin(x/n)}{x/n} \frac{1}{1 + x^2} = \frac{1}{1 + x^2}.$$ 

Hence by the Lebesgue Dominant Convergence Theorem, we have

$$\lim_{n \to \infty} \int_0^n \frac{n}{x(1+x^2)} \sin \left(\frac{x}{n}\right) dx = \int_0^\infty \lim_{n \to \infty} f_n(x) dx = \int_0^\infty \frac{1}{1 + x^2} dx = \arctan x|_0^\infty = \frac{\pi}{4}.$$ 

4. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume all $f, f_n \in L^1(\mu)$ ($n = 1, 2, \ldots$) and $f_n \to f$ in $L^1(\mu)$. Prove that $f_n \to f$ in measure.

**Proof.** Let $\varepsilon > 0$ and $E_n = \{|f_n - f| \geq \varepsilon\}$ ($n = 1, 2, \ldots$). We have

$$\int_X |f_n - f| d\mu \geq \int_{E_n} |f_n - f| d\mu \geq \int_{E_n} \varepsilon d\mu = \varepsilon \mu(E_n).$$

Thus,

$$\mu(E_n) \leq \frac{1}{\varepsilon} \int_X |f_n - f| d\mu \to 0,$$

as $n \to \infty$, since $f_n \to f$ in $L^1(\mu)$. Hence $f_n \to f$ in measure. **Q.E.D.**
5. (20 points) Let $(X, M, \mu)$ be a measure space. Let $f : X \to [0, \infty)$ be a measurable function.

(1) Prove that $f \in L^1(\mu)$ if and only if $\sum_{n=1}^{\infty} n \mu(\{n-1 < f \leq n\}) < \infty$.

(2) Prove that if $f \in L^1(\mu)$ then $\lim_{N \to \infty} N \mu(\{f \geq N\}) = 0$.

Note. If $\mu(X) < \infty$ the proof below is also correct.

Proof. (1) Suppose $f \in L^1(\mu)$. Let $E_0 = \{f = 0\}$ and $E_n = \{n-1 < f \leq n\}$ $(n \in \mathbb{N})$. Then $X$ is the disjoint union of $E_0, E_1, \ldots$ and moreover $1 = \chi_X = \chi_{\cup_{n=0}^{\infty} E_n} = \sum_{n=0}^{\infty} \chi_{E_n}$ on $X$. Consequently, since $f = 0$ on $E_0$ and $f > n - 1$ on $E_n$, we have

\[
\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} (n-1) \, d\mu = \sum_{n=1}^{\infty} n \mu(E_n).
\]

But $(n-1)/n \to 1$ as $n \to \infty$. Thus the convergence of $\sum_{n=1}^{\infty} (n-1) \mu(E_n)$ is equivalent to the convergence of $\sum_{n=1}^{\infty} n \mu(E_n)$. Hence, $\sum_{n=1}^{\infty} n \mu(\{n-1 < f \leq n\}) < \infty$.

Conversely, assume $\sum_{n=1}^{\infty} n \mu(\{n-1 < f \leq n\}) < \infty$. Then, as before, and by the fact that $f \leq n$ on $E_n$ for each $n \geq 1$, we have

\[
\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu \leq \sum_{n=1}^{\infty} \int_{E_n} n \, d\mu = \sum_{n=1}^{\infty} n \mu(E_n) < \infty.
\]

Hence $f \in L^1(\mu)$.

(2) By Part (1), $\sum_{n=1}^{\infty} n \mu(\{n-1 < f \leq n\}) < \infty$. Hence, $\sum_{n=N+1}^{\infty} n \mu(\{n-1 < f \leq n\}) \to 0$ as $N \to \infty$. But

\[
\sum_{n=N+1}^{\infty} n \mu(\{n-1 < f \leq n\}) \geq \sum_{n=N+1}^{\infty} N \mu(\{n-1 < f \leq n\}) = N \mu(\cup_{n=N+1}^{\infty} \{n-1 < f \leq n\}) = N \mu(\{f \geq N\}).
\]

Hence $\lim_{N \to \infty} N \mu(\{f \geq N\}) = 0$. Q.E.D.