1. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of sets. Define

$$\limsup_{n \to \infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\},$$

$$\liminf_{n \to \infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}.$$ 

Prove that

$$\limsup_{n \to \infty} E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \quad \text{and} \quad \liminf_{n \to \infty} E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n.$$ 

2. Let $X$ and $Y$ be two sets and $f : X \to Y$ a mapping. Let $\{Y_\alpha\}_{\alpha \in A}$ be a family of subsets of $Y$. Prove

$$f^{-1}\left(\bigcup_{\alpha \in A} Y_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(Y_\alpha) \quad \text{and} \quad f^{-1}\left(\bigcap_{\alpha \in A} Y_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(Y_\alpha).$$

3. Find a bijection from $\mathbb{N}$ to $\mathbb{N}^2$.

4. Construct a sequence of open sets $U_n (n = 1, 2, \ldots)$ of $\mathbb{R}$ such that $\cap_{n=1}^{\infty} U_n$ is not open.

5. Let $X$ be a complete metric space and $E$ a non-empty subset of $X$. Prove that $E$ is closed if and only if $E$ is complete.

6. Let $(Y, B)$ be a measurable space and $X$ a nonempty set. For any $f : X \to Y$, define $\mathcal{A} = \{f^{-1}(B) : B \in B\}$. Prove that $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$.

7. An algebra $\mathcal{A}$ is a $\sigma$-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $E_n \in \mathcal{A}$ for all $n = 1, 2, \ldots$ and $E_1 \subseteq E_2 \subseteq \cdots$, then $\cup_{n=1}^{\infty} E_n \in \mathcal{A}$.)

8. Does there exist an infinite $\sigma$-algebra which has only countably many members? If yes, provide an example. If no, prove it.

9. Let $X$ be a nonempty set and $\mathcal{E}$ a class of subsets of $X$. Let $\mathcal{M}$ be the $\sigma$-algebra of subsets of $X$ generated by $\mathcal{E}$. Prove that $\mathcal{M}$ is the union of the $\sigma$-algebra generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.