1. Let \((X, \mathcal{M}, \mu)\) be a measure space. If \(E, F \in \mathcal{M}\) then \(\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)\).

2. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(E \in \mathcal{M}\). Define \(\mu_E(A) = \mu(A \cap E)\) for any \(A \in \mathcal{M}\). Show that \(\mu_E\) is a measure.

3. Let \((X, \mathcal{M}, \mu)\) be a measure space and \(E_n \in \mathcal{M} (n = 1, 2, \ldots)\). Prove the following:
   
   (1) \(\mu(\lim \inf_{n \to \infty} E_n) \leq \lim \inf_{n \to \infty} \mu(E_n)\);
   
   (2) \(\mu(\lim \sup_{n \to \infty} E_n) \geq \lim \sup_{n \to \infty} \mu(E_n)\), provided that \(\mu(\bigcup_{n=1}^{\infty} E_n) < \infty\).

4. Let \((X, \mathcal{M})\) be a measurable space and \(\mu : \mathcal{M} \to [0, \infty]\) be such that \(\mu(\emptyset) = 0\) and \(\mu\) is finitely additive.
   
   (1) Prove that \(\mu\) is a measure if and only if it is continuous from below as in Theorem 1.8 (c) of the textbook.
   
   (2) Assume in addition that \(\mu(X) < \infty\). Prove that \(\mu\) is a measure if and only if it is continuous from above as in Theorem 1.8 (d) of the textbook.

5. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \(E_n \in \mathcal{M} (n = 1, 2, \ldots)\) satisfy \(\sum_{n=1}^{\infty} \mu(E_n) < \infty\). Prove that \(\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)\) if and only if \(\mu(E_i \cap E_j) = 0\) for any \(i, j\) with \(i \neq j\).

6. If \(\mu\) is a \(\sigma\)-finite measure on a measure space \((X, \mathcal{M})\), then it is semifinite, i.e., for any \(E \in \mathcal{M}\) with \(\mu(E) = \infty\), there exists \(F \in \mathcal{M}\) such that \(F \subseteq E\) and \(0 < \mu(F) < \infty\).

7. Let \(\mu\) be a semifinite measure on a measurable space \((X, \mathcal{M})\). Suppose \(E \in \mathcal{M}\) and \(\mu(E) = \infty\). Show that for any \(C > 0\) there exists \(F \in \mathcal{M}\) such that \(F \subseteq E\) and \(C < \mu(F) < \infty\).

8. Let \(\mu^*\) be an outer measure on \(X\). Let \(\{A_n\}_{n=1}^{\infty}\) be a sequence of disjoint \(\mu^*\)-measurable sets. Prove that \(\mu^*(E \cap (\bigcup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n)\) for any \(E \subseteq X\).

9. Let \(\mathcal{A}\) be an algebra of a set \(X\). Denote by \(\mathcal{A}_\sigma\) the collection of countable unions of sets in \(\mathcal{A}\) and by \(\mathcal{A}_{\sigma\delta}\) the collection of countable intersections of sets in \(\mathcal{A}_\sigma\). Let \(\mu_0\) be a premeasure on \(\mathcal{A}\) and \(\mu^*\) the induced outer measure on \(X\). Prove the following:
   
   (1) For any \(E \subseteq X\) and \(\epsilon > 0\), there exists \(A \in \mathcal{A}_\sigma\) such that \(E \subseteq A\) and \(\mu^*(A) \leq \mu^*(E) + \epsilon\);
   
   (2) If \(E \subseteq X\) and \(\mu^*(E) < \infty\), then \(E\) is \(\mu^*\)-measurable if and only if there exists a \(B \in \mathcal{A}_{\sigma\delta}\) such that \(E \subseteq B\) and \(\mu^*(B \setminus E) = 0\).

10. Let \((X, \mathcal{M}, \mu)\) be a finite measure space. Let \(\mu^*\) be the outer measure induced by \(\mu\). Suppose \(E \subseteq X\) satisfies \(\mu^*(E) = \mu^*(X)\) (but not that \(E \in \mathcal{M}\)).
    
    (1) If \(A, B \in \mathcal{M}\) and \(A \cap E = B \cap E\), then \(\mu(A) = \mu(B)\).
    
    (2) Let \(\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}\). Define \(\nu : \mathcal{M}_E \to [0, \infty]\) by \(\nu(A \cap E) = \mu(A)\) (which makes sense by Part (1)). Then \(\mathcal{M}_E\) is a \(\sigma\)-algebra on \(E\) and \(\nu\) is a measure on \(\mathcal{M}_E\).