Math 240A, Fall 2019
Solution to Problems of HW#3
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1. We have
   \[ \mu \left( \bigcap_{j=1}^{n} A_j \right) = \mu \left( \left( \bigcup_{j=1}^{n} A_j^c \right)^c \right) = 1 - \mu \left( \bigcup_{j=1}^{n} A_j^c \right) 
   = 1 - \sum_{j=1}^{n} \mu(A_j^c) = 1 - \sum_{j=1}^{n} (1 - \mu(A_j)) 
   = 1 - n + \sum_{j=1}^{n} \mu(A_j) > 1 - n + n - 1 = 0. \square \]

2. We have \( E = \bigcap_{n=1}^{\infty} E_n \). The sequence \( E_{n+1} \subseteq E_n \), and \( \mu(E_n) \) decreases with \( n \), and \( \mu \left( \bigcup_{n=1}^{\infty} E_n \right) < \infty \). By the continuity from above of a measure, we have
   \[ \mu(E) = \lim_{n \to \infty} \mu \left( \bigcup_{k=1}^{n} E_k \right). \]
   But, \( \mu \left( \bigcup_{k=1}^{n} E_k \right) \leq \sum_{k=1}^{n} \mu(E_k) \to 0 \) as \( \sum_{k=1}^{\infty} \mu(E_k) < \infty \).
   Thus, \( \mu(E) = 0. \square \]

3. No. Let \( \mathbb{Q} = \{ q_1, q_2, \ldots \} \) be the set of all rational numbers. Then \( G = \bigcup_{k=1}^{\infty} (r_k - \frac{1}{2k}, r_k + \frac{1}{2k}) \) is open and \( \mu(G) \leq \sum_{k=1}^{\infty} 2 \cdot \frac{1}{2k} = 2. \) But \( G = \mathbb{Q} \) and \( \mu(G) = \infty. \square \]

4. We have \( m(BVA) = m(B) - m(A) = 0 \) and \( EVA \subseteq BVA \). Since \( m \) is complete, we have \( EVA \subseteq \mathcal{L} \) (i.e., \( EVA \) is Lebesgue measurable) and \( m(EVA) = 0 \). Now, \( E = (EVA)VA \) is also Lebesgue measurable. \( m(E) = m(EVA)m(A) = m(A). \square \)
5. (1) We have \((-\infty, 1) = \bigcup_{n=1}^{\infty} (-n, 1 - \frac{1}{2^n})\), a union of increasing sets (\(n\)-intervals). Hence,

\[
\mu((-\infty, 1)) = \mu\left(\bigcup_{n=1}^{\infty} (-n, 1 - \frac{1}{2^n})\right)
= \lim_{n \to \infty} \mu\left(-n, 1 - \frac{1}{2^n}\right)
= \lim_{n \to \infty} \left[F\left(1 - \frac{1}{2^n}\right) - F(-n)\right]
= 1 - 0 = 1.
\]

(2) Similarly,

\[
\mu((-\infty, 1]) = \lim_{n \to \infty} \mu\left(-n, 1\right]
= \lim_{n \to \infty} \left[F(1) - F(-n)\right]
= 4 - 0
= 4.
\]

(3) \(\mu([1, \infty)) = \mu((-\infty, 1]) + \mu((1, 2]) + \mu((2, \infty))\)

\[
= 4 + F(2) - F(1) + \lim_{n \to \infty} \mu\left(2, n+2\right]
= 4 + 7 - 4 + \lim_{n \to \infty} \left[F(n+2) - F(2)\right]
= 7 + 7 - 7 = 7.
\]

(4) \(\mu\left\{\{2\}\right\} = \mu\left(\bigcup_{n=1}^{\infty} (2 - \frac{1}{n}, 2 + \frac{1}{n})\right)\)

\[
= \lim_{n \to \infty} \mu\left((2 - \frac{1}{n}, 2 + \frac{1}{n})\right)
= \lim_{n \to \infty} \left[F(2 + \frac{1}{n}) - F(2 - \frac{1}{n})\right]
= 7 - 7 = 0.
\]
Let \( H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases} \) (This is called the Heaviside function.) \( H \) is increasing and right continuous. We show that \( M_H = \delta \). Hence, all increasing and right continuous functions \( F \) with \( M_F = \delta \) are just \( H(x) + c \) for some constants \( c \).

Recall the Dirac measure \( \delta \) is defined by

\[
\delta(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E \end{cases}
\]

for any \( E \in \mathbb{B} \). We have

\[
\delta(0^0) = 1, \quad \delta(1^0) = 1.
\]

\[
M_H(0^0) = \lim_{n \to 0} M_H\left(-\frac{1}{n}, \frac{1}{n}\right) = \lim_{n \to 0} \left[ H\left(\frac{1}{n}\right) - H\left(-\frac{1}{n}\right)\right] = 1,
\]

and

\[
M_H(1^0) = \lim_{n \to 0} M_H\left(0, \frac{1}{n}\right) = \lim_{n \to 0} \left[ M_H\left(\frac{1}{n}\right) - M_H(0)\right] = 1.
\]

Hence, both \( \delta \) and \( M_H \) are concentrated on \( \{0\} \).

Finally, we show that

\[
\delta((a, b]) = H(b) - H(a), \quad \forall a, b \in \mathbb{R}, \ a < b.
\]

\[
\delta((a, b]) = 1 \text{ or } 0 \text{ if } 0 \in (a, b) \text{ or } 0 \notin (a, b).
\]

Same for the right-hand side: \( H(b) - H(a) = M_H((a, b]) = 1 \text{ or } 0 \text{ if } 0 \in (a, b) \text{ or } 0 \notin (a, b) \).

As \( M_H(1^0) = 1 \).

\( \square \)
Proposition 1.20. If \( E \in \mathcal{M}_\mu \) and \( \mu(E) < \infty \), then for any \( \varepsilon > 0 \) there exists a set \( A \) that is a finite union of open intervals such that \( \mu(E \Delta A) < \varepsilon \).

Let \( \varepsilon > 0 \). By Theorem 1.18, there exists an open set \( U \supseteq E \) such that \( \mu(U) < \mu(E) + \frac{\varepsilon}{2} \). If \( U \) is already a finite union of open intervals, then let \( A = U \).

We have:

\[
\mu(E \Delta A) = \mu((E \setminus A) \cup (A \setminus E)) \\
\leq \mu(E \setminus A) + \mu(A \setminus E) \\
= \mu(A) - \mu(E) < \frac{\varepsilon}{2} < \varepsilon.
\]

Otherwise, \( U = \bigcup_{j=1}^{\infty} (a_j, b_j) \) for some \( a_j, b_j \in \mathbb{R} \) with \( a_j < b_j \) (\( j = 1, 2, \ldots \)). [Note that any open set in \( \mathbb{R} \) is a countable union of open intervals.]

Hence,

\[
\mu(U) = \sum_{j=1}^{\infty} \mu((a_j, b_j)) < \mu(E) + \frac{\varepsilon}{2} < \infty.
\]

There exists \( N \in \mathbb{N} \) such that

\[
\sum_{n=N+1}^{\infty} \mu((a_j, b_j)) < \frac{\varepsilon}{2}.
\]

Let \( A = \bigcup_{n=1}^{N} (a_j, b_j) \subseteq U \). \( A \) is a finite union of open intervals. \( \mu(U \setminus A) = \sum_{n=N+1}^{\infty} \mu((a_j, b_j)) \)

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

\[\Box\]
Let $\varepsilon = \frac{1}{2^n} \ (0,1)$ and $\varepsilon_k = \varepsilon^n / 2^{k-1} \ (k=1,2,\ldots)$

Divide $[0,1]$ into

3 intervals with

the middle open

one having length $\varepsilon_1$ and the other two closed intervals

having same length. Denote by $F_1$ the union of the

two closed intervals.

For each of the two closed intervals of $F_1$, divide it

into 3 intervals with the middle open interval

of length $\varepsilon_2$ and the other two closed intervals

having same length. Denote by $F_2$ the 4 closed

remaining disjoint intervals.

Continuing by induction, we have a sequence of

closed sets $F_k \ (k=1,2,\ldots)$. Each $F_k$ comes is the

union of $2^k$ disjoint closed intervals of same

length.

Let $F = \bigcup F_k$. Then $F \subseteq [0,1]$. $F$ is closed and

hence compact.

If $0 < a < b < 1$ and $(a,b) \subseteq F$. Then $(a,b) \subseteq F_k \ (k=1,2,\ldots)$

But $F_k$ is the union of $2^k$ disjoint closed intervals

So, $(a,b)$ is contained in one such interval which

has the length $\leq \frac{1}{2^k}$ as $m(F_k) \leq 1$. But as $k$ large

enough $b-a \geq \frac{1}{2^k}$. This is impossible. Thus $F$

contains no open interval. Hence $F$ is nowhere

closed.

Finally $\mu(F) = 1 - \varepsilon_1 - 2\varepsilon_2 - \varepsilon_3 - \ldots = 1 - 2 - \varepsilon^n - \ldots$

$= 1 - \frac{\varepsilon}{1-\varepsilon} = 1 - \frac{\varepsilon}{2-\varepsilon} / 1 - \frac{\varepsilon}{2-\varepsilon} = \varepsilon \quad \square$
9. If the statement were not true, then there exists \( x \in (0,1) \) such that \( m(E \cap I) \geq x m(I) \) for any open interval \( I \).

\[ \forall \varepsilon > 0. \exists \text{ open set } U \supseteq E \text{ such that } m(U) < m(E) + \varepsilon. \] Let \( U = \bigcup_{i=1}^{\infty} I_i \) with each \( I_i \) an open interval and \( I_i \cap I_j = \emptyset \) if \( i \neq j \). (The case that \( U \) is a finite union of disjoint open intervals can be treated similarly.) Since \( m(E \cap I_i) \leq m(I_i) \), we have for each \( i \)

\[ m(U) \leq m(\bigcup_{i=1}^{\infty} I_i) = m(U \setminus E) + m(U \cap E) \leq m(U \setminus E) + \varepsilon. \]

and hence \( m(U) = \frac{1}{1-x} m(U \setminus E) \).

Consequently,

\[ 0 < m(E) \leq m(U) = \sum_{i=1}^{\infty} m(I_i) \leq \frac{1}{1-x} \sum_{i=1}^{\infty} m(I_i \setminus E) \]

\[ = \frac{1}{1-x} m\left( \bigcup_{i \geq 1} (I_i \setminus E) \right) = \frac{1}{1-x} m(U \setminus E) < \frac{\varepsilon}{1-x}. \]

Since \( \varepsilon > 0 \) is arbitrary, this implies \( m(E) = 0 \), a contradiction. \( \Box \)

10. (1) Recall that \( N_r = (N \cap [0,1-r]) + r \cup (N \cap [1-r,1]) - (1-r) \) for each \( r \in \mathbb{Q} \cap [0,1) \). Define similarly

\[ E_r = (E \cap [0,1-r]) + r \cup (E \cap [1-r,1]) - (1-r) \subseteq N_r, \forall r \in \mathbb{Q} \cap [0,1) \]

Since \( N_r \cap N_s = \emptyset \) if \( r \neq s \), \( r, s \in \mathbb{Q} \cap [0,1) \), we have \( E_r \cap E_s = \emptyset \) if \( r \neq s \), \( r, s \in \mathbb{Q} \cap [0,1) \). This is clear by the translation invariance of the Lebesgue measure that \( m(E_r) = m(E) \). Thus, if \( m(E) > 0 \) then

\[ m(E) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(E_r) = \sum_{r \in \mathbb{Q} \cap [0,1)} m(U \setminus E_r) \leq m([0,1)) = 1. \]

a contradiction. Hence \( m(E) = 0 \).
(2) Note that \( E = \bigcup_{n \in \mathbb{Z}} (E \cap [n, n+1]) \), a disjoint union.

Since \( m(E) > 0 \), there exists \( n \in \mathbb{Z} \) such that \( m(E \cap [n, n+1]) > 0 \). Let \( F = E \cap [n, n+1] \) and \( m(F) > 0 \). If we can show that \( F \) contains a Lebesgue non-measurable subset, then \( E \cap [n, n+1] \) contains a Lebesgue non-measurable set \( \mathcal{D} + n \).

So, it suffices to assume \( E \subseteq [0, 1) \).

Suppose \( m(E) > 0 \) but any subset of \( E \) is Lebesgue measurable.

Observe that for Part (1) holds true with \( N \) replaced by \( \mathcal{N}_r \) for any \( r \in (0, 1) \), since \( \mathcal{N}_r \) consists points exactly one from that each equivalence class defined by \( x \sim y \Leftrightarrow x - y \in \mathbb{Q} \). Thus, \( m(\mathcal{N}_r) = 0 \)

Consequently, since \( [0, 1) = \bigcup_{r \in (0, 1)} \mathcal{N}_r \) disjoint, we get

\[
0 \leq m(E) = m(E \cap [0, 1)) = m\left( \bigcup_{r \in (0, 1)} (E \cap \mathcal{N}_r) \right) = \sum_{r \in (0, 1)} m(E \cap \mathcal{N}_r) = 0,
\]

This is a contradiction. Hence, \( E \) contains a non-measurable set. \( \square \)