1. (1) Assume $f$ is measurable. Then for any $r \in \mathbb{R}$, $(r, \infty) \in \mathcal{B}_R$. Hence $f^{-1}(r, \infty) \in \mathcal{M}$.

   Conversely, assume $f^{-1}(r, \infty) \in \mathcal{M}$ for all $r \in \mathbb{R}$. Let $a \in \mathbb{R}$. Choose $r_n \in \mathbb{R}$ such that $r_n$ decreases and $r_n \to a$. Then, $(a, \infty) = \bigcap_{n=1}^{\infty} (r_n, \infty)$ and $f^{-1}((a, \infty)) = \bigcap_{n=1}^{\infty} f^{-1}((r_n, \infty)) \in \mathcal{M}$ since each $f^{-1}((r_n, \infty)) \in \mathcal{M}$. But $\{a, \infty\}$ generates $\mathcal{B}_R$. Hence, $f$ is measurable.

2. Let $F(x) = \liminf_{n \to \infty} f_n(x)$ and $\overline{f}(x) = \limsup_{n \to \infty} f_n(x)$ for any $x \in X$. Then $F(x) \geq f(x)$ $(x \in X)$. Both $F$ and $\overline{f}$ are measurable. Thus

   $\{x \in X : \liminf_{n \to \infty} f_n(x) \text{ exists} \} = \{x \in X : F(x) = f(x) \in (0, \infty) \}$

   $= \{x \in X : (\overline{f} - f)(x) = 0 \} = (\overline{f} - f)^{-1}(\{0\}) \in \mathcal{M}$

   as $\{0\} \in \mathcal{B}_R$ and $\overline{f} - f$ is measurable.

2. Assume $f$ is increasing. (The case that $f$ is decreasing can be treated similarly.)

   Let $x \in \mathbb{R}$ and define $A_x = \{x \in \mathbb{R} : f(x) > x\}$. If $A_x = \emptyset$ then $A_x \in \mathcal{B}_R$. So, assume $A_x \neq \emptyset$. Set $x_0 = \inf A_x$. If $x_0 = -\infty$ then $A_x = (-\infty, 0) \in \mathcal{B}_R$.

   Indeed, let $x \in \mathbb{R}$, then since $x_0 = -\infty$ there exists $\rho \in A_x$ such that $\rho < x$. Then
\[ f(x) \geq f(x_0) > x. \] So, \( x \in Ax \) and \( Ax \subseteq \mathbb{R} \). But if \( Ax \subseteq \mathbb{R} \) then \( Ax = \mathbb{R} \subseteq \mathbb{R} \). Finally, assume \( x_0 = m(f \cap Ax \cap (-\infty, x)) > -\infty \). Then, clearly, \( x < x_0 \Rightarrow x \notin Ax \). Moreover, if \( x > x_0 \) then \( \exists x_i \in Ax \) such that \( x_0 < x_i < x \). Hence \( f(x) \geq f(x_i) > x \) and \( x \notin Ax \). Therefore \( (-\infty, x_0) \cap Ax = \emptyset \) and \( Ax = (x_0, \infty) \). Consequently, either \( Ax = (x_0, \infty) \) or \( Ax = [x_0, \infty) \). In both cases, \( Ax \in \mathcal{B}_{\mathbb{R}} \). Thus \( f \) is Borel measurable.

4. Let \( f(x) = X_{(x_0, x)}(x), x \in \mathbb{R} \). Then \( f \) is Lebesgue measurable since \( f'(x) = 0 \) a.e. But \( f \) is nowhere continuous as \( f = 1 \) at \( x_0 \in \mathbb{Q} \) and \( 0 \) at \( x \in \mathbb{R} \), and \( \mathbb{Q} \) is dense in \( \mathbb{R} \), and \( \mathbb{Q}^c \) is also dense in \( \mathbb{R} \), as \( m(\mathbb{Q}^c \cap (a, b)) = b - a \) for any \( a < b \).

3. Recall that \( f : X \rightarrow \mathbb{R} \) is measurable on \( A \subseteq \mathcal{M} \) means that for any \( E \in \mathcal{B}_{\mathbb{R}} \), \( f^{-1}(E) \in \mathcal{M} \). If \( f \) is measurable then \( f^{-1}(E) \in \mathcal{M} \) for any \( E \in \mathcal{B}_{\mathbb{R}} \). If \( A, B \subseteq \mathcal{M} \), then \( f^{-1}(E) \cap A \subseteq \mathcal{M} \) and \( f^{-1}(E) \cap B \subseteq \mathcal{M} \). Thus \( f \) is measurable on \( A \) and on \( B \).

Conversely, suppose \( X = A \cup B \), \( A, B \subseteq \mathcal{M} \), and \( f : X \rightarrow \mathbb{R} \) is measurable on \( A \) and on \( B \). Let
5. No. Example. \( \mu = \delta \) : the Dirac mass concentrated on \( \{0\} \), i.e., \( \delta(E) = 1 \) if \( 0 \in E \), \( \delta(E) = 0 \) if \( 0 \notin E \) where \( E \in B \mathbb{R} \) Let \( V = (0, 1) \). Then

\[
f(x) = \delta((0, 1) + x) = \delta((x, x+1))
\]

\[
= \begin{cases} 
1 & \text{if } x < 0 < x+1, \text{ i.e. } -1 < x < 0 \\
0 & \text{otherwise}
\end{cases}
\]

Clearly, \( f(x) = \chi_{(-1, 0)}(x) \) it is discontinuous at \( x = 0, -1 \). □

6. (1) Note that \( f: [0, 1] \to [0, 1] \) is continuous, nondecreasing, and \( f([0,1]) = [0,1] \).

Clearly \( g(x) = f(x) + x \) is strictly increasing on \([0,1] \). Hence \( g \) is injective on \([0,1] \). Moreover, \( g(0) = 0 \), \( g(1) = f(1) + 1 = 2 \). By the Intermediate Value Theorem, for any \( z \in (0, 2) \), \( z \in (0,1) \) such that \( g(z) = z \), since \( g \) is continuous on \([0,1] \). Thus, \( g: [0,1] \to [0,2] \) is surjective. Hence, \( g: [0,1] \to [0,2] \) is a bijection. Consequently, \( h = g^{-1} : [0, 2] \to [0,1] \) is continuous as \( G \) is continuous. [In general, if \( g: [a, b] \to [a,b] \) is strictly increasing, continuous, and bijective, then \( g: ([a,b]) \to [a,1] \) is continuous.]
Otherwise, \(\exists x_0 \in [a,b], \exists x_n \in [a,b]\) such that \(x_n \to x_0\). But \(g^{-1}(x_n) \neq g^{-1}(x_0)\). Without loss of generality, we may assume that \(x_0\) such that
\[
g^{-1}(x_n) > n + g^{-1}(x_0) \quad (n=1, 2, \ldots)
\]
Thus, \(x_n > g(n + g^{-1}(x_0)) > g(g^{-1}(x_0)) = x_0\). Hence, \(x_n \to x_0\), a contradiction.

(2) Note that the Cantor function \(f: [0,1] \to [0,1]\) is constant on any interval of \([0,1] \setminus C\). If \(I\) is such an interval, then \(g\) translates \(I\) by the constant, and \(m(g(I)) = m(I)\). But the closed set \([0,1] \setminus C\) is a countable union of disjoint such intervals. Thus, \(m(g([0,1] \setminus C)) = m([0,1] \setminus C) = 1\). By Part (1), \(m(g(C)) + m(g([0,1] \setminus C)) = m([0,2]) = 2\). Hence, \(m(g(C)) = 1\).

(3) We have \(g(B) = A \subseteq g(C)\). Hence \(B \subseteq C\).
But \(m(C) = 0\), and \(m\) is complete. Hence \(B\) is Lebesgue measurable, and \(m(B) = 0\).
If \(B\) were Borel measurable, \(A = g^{-1}(B)\) would be also Borel measurable, since \(g^{-1}\) is continuous by Part (1). Hence, \(A\) would be Lebesgue measurable, a contradiction.

(4) Let \(F = B\), as in Part (3). This is Lebesgue measurable, since \(B\) is. Let \(G = g^{-1}\) as above. Then \((F \circ G)(F((B))) = g^{-1}(B) = g(B) = A\). Since \(A\) is non-Lebesgue measurable, \(F \circ G\) is not Lebesgue measurable.
7. Suppose there existed $x_0 \in (0, 1)$ such that $f(x_0) \neq g(x_0)$, say, $f(x_0) > g(x_0)$. Let $\lambda = f(x_0)$. Since $f$ is decreasing, \( \{ f \geq \lambda \} = \{ f \geq f(x_0) \} \supseteq (0, x_0] \), and since $g$ is decreasing and left continuous, \( \exists x_1 : 0 < x_1 < x_0, \) such that $f(x_0) > g(x_1) \geq g(x_0)$. Thus, \( \{ g \geq \lambda \} = \{ g \geq f(x_0) \} \supseteq \{ g > g(x_1) \} \subseteq (0, x_1] \). Therefore, \( m(\{ g \geq \lambda \}) \geq m((0, x_0]) = x_0 \)

\( m(\{ g > g(x_1) \}) \leq m((0, x_1]) = x_1 < x_0 = m(\{ f \geq \lambda \}) \). This is a contradiction. Hence $f = g$ on $(0, 1)$. \( \square \)

8. \( \lambda(\emptyset) = \int X \phi f \, du = 0 \).

If $E_j \in \mathcal{P}$ ($j = 1, 2, \ldots$) are disjoint, then \( \lambda \left( \bigcup_{j=1}^{\infty} E_j \right) = \int f \, du = \int \chi_{E_1} f \, du = \int \chi_{E_2} f \, du = \ldots \int \chi_{E_j} f \, du = \ldots \int \chi_{E_{\infty}} f \, du = \lim_{j \to \infty} \int \chi_{E_j} f \, du \). Hence $\lambda$ is a measure.

If $\phi \in L^+$ is a simple function with $\phi = \sum_{j=1}^{m} a_j \chi_{E_j}$, $a_j > 0$.

If $E_j \in \mathcal{P}$, disjoint, \( \bigcup_{j=1}^{\infty} E_j = X \). Then

\[ \int_X \phi \, du = \sum_{j=1}^{m} a_j \int_{E_j} f \, du = \sum_{j=1}^{m} a_j \int_{E_j} f \, du = \sum_{j=1}^{m} a_j \int_{E_j} f \, du = \sum_{j=1}^{m} a_j \int_{E_j} f \, du = \int_X \phi f \, du. \]

Let $g \in L^+$. Let \( \{ \phi_n \} \) be a sequence of increasing simple functions in $L^+$ such that $\phi_n \to g$. Then, $0 \leq \phi_1 \leq \phi_2 \leq \ldots$ and $\phi_n f \to gf$. By the Monotone Convergence Theorem,

\[ \int_g \, du = \lim_{n \to \infty} \int_{\phi_n} f \, du = \lim_{n \to \infty} \int_{\phi_n} f \, du = \sum_{j=1}^{m} a_j \int_{E_j} f \, du. \]
9. (1) Denote \( E_0 = \{ f = 0 \} \) and \( E = \{ f > 0 \} \). For any \( n \in \mathbb{N} \),
\[
\infty > \int_{E_0} f \, du = \int_{E_0} f \, du > \int_{E_0} ndu = n \mu(E_0).
\]
\[
\sum_{n=1}^{\infty} \mu(E_0) = 1 \int_{E_0} f \, du \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence \( \mu(E_0) = 0 \).

Let \( F = \{ f > 1 \} \) and \( F_n = \{ \frac{1}{n+1} \leq f < \frac{1}{n} \} \) \((n=1, 2, \ldots)\). Then \( X = E_0 \cup F \cup \bigcup_{n=1}^{\infty} F_n \). This is a disjoint union of measurable sets. By Part (1), we have
\[
\infty > \int_{F} f \, du = \int_{E_0} f \, du + \int_{F_0} f \, du = \int_{X} f \, du - \int_{E_0} f \, du
\]
\[
= \int_{F} f \, du + \sum_{n=1}^{\infty} \int_{F_n} f \, du \geq \int_{F} f \, du + \sum_{n=1}^{\infty} \int_{F_n} f \, du
\]
\[
= \mu(F) + \sum_{n=1}^{\infty} \frac{1}{n+1} \mu(F_n).
\]
Thus, \( \mu(F) < \infty, \mu(F_n) < \infty \) \((n=1, 2, \ldots)\). Hence \( \mu \) is \( \sigma \)-finite.

(2) \( \forall \varepsilon > 0 \). Continuing from the above, we have
\[
\int_{X} f \, du = \int_{F} f \, du + \sum_{n=1}^{\infty} \int_{F_n} f \, du < \infty
\]
Thus, \( \exists N \in \mathbb{N} \) such that
\[
\int_{F} f \, du - \varepsilon < \int_{F} f \, du + \sum_{n=1}^{N} \int_{F_n} f \, du = \int_{E} f \, du
\]
where \( E = F \cup \bigcup_{n=1}^{N} (F_n) \subseteq F \), and \( \mu(E) \leq \mu(F) + \sum_{n=1}^{N} \mu(F_n) \lesssim \varepsilon \) as shown above. \( \square \)
10. We show (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1),

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (2),

(4) $\Rightarrow$ (3), (1) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2). We have $|f| \in L^1$ and $|f| = 0$ a.e.

If $\psi \in L^1$ is a simple function and $\phi = 0$ a.e. Then by definition we have $\int \phi \, du = 0$.

If $\phi$ is a simple function, $0 \leq \phi \leq |f|$, then $\phi = 0$ a.e.

Hence $\int \phi \, du = 0$. Thus,

$$\int |f| \, du = \sup \{ \int \phi \, du : 0 \leq \phi \leq |f|, \phi \text{ simple} \}$$

$$= 0.$$

(2) $\Rightarrow$ (1). By the disjoint union of measurable sets

$$X = \{ |f| = 0 \} \cup \{ |f| \geq 1 \} \cup \bigcup_{n=1}^{\infty} \{ \frac{1}{n+1} \leq |f| < \frac{1}{n} \},$$

we have

$$0 = \int |f| \, du = \int_{|f| = 0} |f| \, du + \int_{|f| \geq 1} |f| \, du$$

$$+ \sum_{n=1}^{\infty} \int_{\{ \frac{1}{n+1} \leq |f| < \frac{1}{n} \}} |f| \, du$$

$$\geq \int_{|f| \geq 1} 1 \, du + \sum_{n=1}^{\infty} \int_{\{ \frac{1}{n+1} \leq |f| < \frac{1}{n} \}} \frac{1}{n+1} \, du$$

$$= \mu\left( \{ |f| \geq 1 \} \right) + \sum_{n=1}^{\infty} \mu\left( \left\{ \frac{1}{n+1} \leq |f| < \frac{1}{n} \right\} \right)$$

$$= \mu\left( \{ |f| > 0 \} \right) \geq 0.$$

Hence $\mu(\{ |f| > 0 \}) = 0$ and $f = 0$ a.e.
(2) ⇒ (3)
∀E ∈ Μ, \| f dμ \| ≤ \| f^+ dμ \| ≤ \| f^- dμ \| = 0.

(3) ⇒ (2): Let E = \{ f ≥ 0 \} ∈ Μ. Then,
0 = \int f dμ = \int f^+ dμ = \int f^- dμ = 0.
Similarly, \int f^- dμ = 0. Hence,
\int |f| dμ = \int (f^+ + f^-) dμ = \int f^+ dμ + \int f^- dμ = 0.

(4) ⇒ (3): Let \varphi = \chi_E \text{ for } E \in Μ. Then
\int f \varphi dμ = \int f dμ = 0.

(1) ⇒ (4): \forall \varphi : X → \mathbb{R}, measurable. Since f = 0 a.e., f \varphi = 0 a.e. Hence, as shown that
(1) ⇒ (2), we have \int |f| dμ = 0. But then
\int |f \varphi| dμ ≤ \int |f| dμ = 0. This implies f \varphi ∈ L^1(μ).