Math 240A: Real Analysis, Fall 2019

Homework Assignment 6
Due Friday, November 15, 2019

1. Let \( f \in L^1(m) \). Assume \( f(0) = 0 \) and \( f'(0) \) exists. Define \( g : \mathbb{R} \to \mathbb{R} \) by \( g(0) = 0 \) and \( g(x) = f(x)/x \) if \( x \neq 0 \). Prove that \( g \in L^1(m) \).

2. (1) Find the smallest \( c \in \mathbb{R} \) such that \( \log(1 + e^t) < c + t \) for all \( t \in (0, \infty) \).
(2) Let \( f : [0,1] \to [0,\infty) \) be Lebesgue integrable. Show that the following limit exists and calculate its value:
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^1 \log \left[ 1 + e^{nf(x)} \right] \, dx.
\]

3. Construct Lebesgue integrable functions \( f_n : [0,1] \to [0,1] \) \((n = 1, 2, \ldots)\) such that \( \lim_{n \to \infty} \int_0^1 f_n \, dm = 1 \) and \( \{f_n(x)\} \) diverges for any \( x \in [0,1] \).

4. Prove the following variant of Egoroff’s theorem: Let \( (X, M, \mu) \) be a measure space. Assume: (1) \( f, f_n : X \to \mathbb{C} \) are all measurable and \( f_n \to f \) a.e.; (2) there exists \( g \in L^1(\mu) \) such that \( |f_n| \leq g \) on \( X \) for all \( n \). Then, for any \( \varepsilon > 0 \), there exists \( E \subseteq M \) such that \( \mu(E) < \varepsilon \) and \( f_n \to f \) uniformly on \( E^c \).

5. Prove Lusin’s Theorem: Let \( -\infty < a < b < \infty \) and \( f : [a,b] \to \mathbb{C} \) be Lebesgue measurable. For any \( \varepsilon > 0 \), there exists a compact set \( E \subseteq [a,b] \) such that \( m(E^c) < \varepsilon \) and \( f|_E \) is continuous.

6. Let \( (X, M, \mu) \) and \( (Y, N, \nu) \) be two measure spaces. Let \( f : X \to \mathbb{C} \) and \( g : Y \to \mathbb{C} \) be two functions and define \( h : X \times Y \to \mathbb{C} \) by \( h(x,y) = f(x)g(y) \) for any \( x \in X \) and \( y \in Y \). Prove the following:
(1) If \( f : X \to \mathbb{C} \) is \( M \)-measurable and \( g : Y \to \mathbb{C} \) is \( N \)-measurable, then \( h : X \times Y \to \mathbb{C} \) is \( M \otimes N \)-measurable;
(2) If \( f \in L^1(\mu) \) and \( g \in L^1(\nu) \), then \( h \in L^1(\mu \times \nu) \) and \( \int_{X \times Y} h \, d(\mu \times \nu) = \left( \int_X f \, d\mu \right) \left( \int_Y g \, d\nu \right) \).

7. Let \( (X, M, \mu) \) be a \( \sigma \)-finite measure space, \( f : X \to [0,\infty) \) a measurable function, and \( G_f = \{(x,y) \in X \times [0,\infty) : y \leq f(x)\} \).

Prove that \( G_f \) is \( M \otimes B_{\mathbb{R}} \)-measurable and that \( (\mu \times m)(G_f) = \int_X f \, d\mu \).

8. Prove
\[
\int_0^1 \int_0^\infty \left( e^{-xy} - 2e^{-2xy} \right) \, dy \, dx 
\neq \int_0^\infty \int_0^1 \left( e^{-xy} - 2e^{-2xy} \right) \, dx \, dy.
\]

9. Use Fubini’s Theorem and the formula \( \frac{1}{x} = \int_0^\infty e^{-xt} \, dt \) \((x > 0)\) to prove \( \lim_{A \to \infty} \int_0^A \frac{\sin x}{x} \, dx = \frac{\pi}{2} \).

10. Let \( a > 0, f : (0,a) \to \mathbb{R} \) be Lebesgue integral on \((0,a), \) and \( g(x) = \int_x^a t^{-1} f(t) \, dt \) \((0 < x < a)\). Prove that \( g \) is integrable on \((0,a)\) and \( \int_{(0,a)} g \, dm = \int_{(0,a)} f \, dm \).