Math 240A, Fall 2019
Solution to Problems of HW #6
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10. Note that $f$ is Lebesgue measurable, and the function $V: \mathbb{R} \to \mathbb{R}$ defined by $V(x) = \frac{1}{x}$ if $x \neq 0$ and $V(0) = 0$, is also measurable. Hence $g(x) = f(x) V(x)$ is also measurable.

Since $f(0) = 0$ and $f(x)$ exists,

$$|g(x)| = \left| \frac{f(x)}{x} \right| = \left| \frac{f(x) - f(0)}{x} \right| \to |f(0)| \text{ as } x \to 0.$$ 

Thus, $\exists \delta > 0$ such that

$$|g(x)| \leq 1 + |f(x)| \quad \forall x \in [-\delta, \delta].$$

(Note that $g(0) = 0$.) Therefore,

$$\int |g| \, dm = \int_{[-\delta, \delta]} |g| \, dm + \int_{\mathbb{R} \setminus [-\delta, \delta]} |g| \, dm$$

$$\leq \int_{[-\delta, \delta]} [1 + |f(0)|] \, dm + \int_{\mathbb{R} \setminus [-\delta, \delta]} \frac{|f(x)|}{|x|} \, dm(x)$$

$$\leq 2\delta [1 + |f(0)|] + \int_{\mathbb{R} \setminus [-\delta, \delta]} \frac{1}{|x|} |f(x)| \, dm(x)$$

$$\leq 2\delta [1 + |f(0)|] + \frac{1}{\delta} \int_{\mathbb{R} \setminus [-\delta, \delta]} |f| \, dm$$

$$< \infty.$$ 

Hence, $g \in L^1(m)$. \qed
2. (1) \( \log (1+e^t) < c + t \quad (\forall t > 0) \)

\[ \Rightarrow \quad 1 + e^t < e^{c+t} = e^c e^t \quad (\forall t > 0). \]

Hence \( c > 0 \). Write \( c = \log a \) for some \( a > 1 \).

\[ \log (1+e^t) < c + t \iff \log (1+e^t) < \log a + t \]

\[ \Rightarrow \quad \frac{\log (1+e^t)}{a} < t \iff \frac{1 + e^t}{a} < e^t \iff 1 \frac{1 + e^t}{a} < e^t \forall t > 0. \]

Let \( t \to \infty \). We get \( t \leq a - 1 \), i.e., \( a \geq 2 \). So, the smallest \( c = \log 2 \).

(2) Let \( g_n(x) = \frac{1}{n} \log [1 + e^{nx}] \quad (x \in [0,1]) \)

\( g(x) = \ln 2 + f(x) \quad (x \in [0,1]). \)

By L'Hopital's rule

\[ \lim_{t \to +\infty} \frac{\ln (1+e^t)}{t} = \lim_{t \to +\infty} \frac{e^t}{1+e^t} = 1. \]

Thus, if \( f(x) > 0 \) for some \( x \in [0,1] \) then

\[ \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \log [1 + e^{nx}] / [n f(x)] = f(x) \]

If \( f(x) = 0 \) at \( x \in [0,1] \) Then

\[ \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \frac{1}{n} \ln 2 = 0 = f(x). \]

Hence \( \lim_{n \to \infty} g_n(x) = f(x) \quad (x \in [0,1]). \)

By Part (1), \( |g_n(x)| \leq \frac{1}{n} (\log 2 + n f(x)) \)

\[ \leq \log 2 + f(x) = g(x) \quad (x \in [0,1]). \]

Since \( f \in L^1([0,1]) \) we have \( g \in L^1([0,1]) \). Thus, it follows from the Lebesgue Dominated Convergence Theorem that

\[ \lim_{n \to \infty} \int_0^1 g_n(x) \, dx = \int_0^1 f(x) \, dx. \]

i.e., \( \lim_{n \to \infty} \int_0^1 \log [1 + e^{nx}] \, dx = \int_0^1 f(x) \, dx. \) \( \square \)
For any \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \) with \( 0 < k \leq 2^n \), define

\[
G_{n,k}(x) = \begin{cases} 
(-1)^n & \text{if } x = 0, \\
\chi(x) & \text{if } 0 < x \leq 1,
\end{cases}
\]

where

\[
I_{n,k} = \left[ \frac{k-1}{2^n}, \frac{k}{2^n} \right].
\]

Note that \((0,1] = \bigcup_{k=1}^\infty I_{n,k}\) is a disjoint union.

Also, \(G_{n,k}(x) = \begin{cases} 1 & \text{if } x \neq 0, x \notin I_{n,k} \\
0 & \text{if } x \in I_{n,k}
\end{cases}\)

Clearly, \(\int_0^1 G_{n,k} \, dm = 1 - m(I_{n,k}) = 1 - \frac{1}{2^n} \to 0\)

as \(n \to \infty\).

Clearly \(G_{n,k}(0) = (-1)^n\) diverges. If \(x \in (0,1]\), then since \((0,1] = \bigcup_{k=1}^\infty I_{n,k}\) for each \(n \in \mathbb{N}\), there exists \(k = k(n,x)\) such that \(x \in I_{n,k}\). Thus, for each \(n \in \mathbb{N}\), \(x\) is such that \(G_{n,k}(x) = 0\).

But, by the same disjoint union, there are other \(k\)'s (in fact all other \(k\)'s) such that \(G_{n,k}(x) = 1\). Thus \(\{G_{n,k}(x)\}\) diverges.

Now, relabel \(G_{n,k}(n=1,2,\ldots, k=1,2,\ldots, 2^n)\) into 
\(\{f_m\}_{m=1}^\infty\) in a natural order (\(n=1,2,\ldots\), for each \(n\), \(k=1,\ldots, 2^n\)) with \(m = 2^{n-1+k}(n=1,2,\ldots, k=0,1,\ldots, 2^n-1)\). Then, \(\int f_m \, dm \to 1\) as \(m \to \infty\) and \(\{f_m(x)\}\) diverges at any \(x \in [0,1]\). \(\square\)
4. Since \( |f_n| \leq G \) in \( X \) and \( f_n \to f \) a.e., we have \( |f| \leq G \) a.e.

Hence \( |f_n - f| \leq 2G \) a.e. If \( k \in \mathbb{N} \) then \( \exists A_n(k) \subset B_k \)

such that \( \mu(A_n(k)) = 0 \) and

\[
\{ |f_n - f| \geq \frac{1}{k} \} \subseteq \{ 2G \geq \frac{1}{k} \} \cup A_n(k), \quad n = 1, 2, ...;
\]

Fix \( k \geq 1 \). Let \( E_n(k) = \bigcup_{n=1}^{\infty} \{ |f_n - f| \geq \frac{1}{k} \} \) \( (n = 1, 2, ...) \).

Then

\[
\lim_{n \to \infty} E_n(k) = \bigcap_{n=1}^{\infty} \{ |f_n - f| \geq \frac{1}{k} \} = \{ x \in X : \text{there are infinitely many } n \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| \geq \frac{1}{k} \}.
\]

Since \( f_n \to f \) a.e., \( \mu(\bigcap_{n=1}^{\infty} E_n(k)) = 0 \).

Note that \( E_n(k) \) decreases as \( n \) increases. Moreover,

\[
E_1(k) = \bigcup_{n=1}^{\infty} \{ |f_n - f| \geq \frac{1}{k} \} \subseteq \bigcup_{n=1}^{\infty} \{ 2G \geq \frac{1}{k} \} \cup A_n(k).
\]

But \( G \in L^1(X) \). So, \( \mu(\{ 2G \geq \frac{1}{k} \}) < \infty \). Also, \( \mu(\bigcup_{n=1}^{\infty} A_n(k)) \leq \sum_{n=1}^{\infty} \mu(A_n(k)) = 0 \). Therefore, \( \mu(E_1(k)) < \infty \).

Consequently, by the continuity of measure from above

\[
\lim_{n \to \infty} \mu(E_n(k)) = \mu(\bigcap_{n=1}^{\infty} E_n(k)) = 0.
\]

(Let \( \varepsilon > 0 \). We can choose \( k_n \uparrow \infty \) and \( \mu(E_{k_n}(x)) < \varepsilon / 2^n \)

\( (k = 1, 2, ...) \). Let \( E = \bigcap_{k=1}^{\infty} E_{k_n}(x) \). Then

\[
\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_{k_n}(x)) < \sum_{k=1}^{\infty} \varepsilon / 2^n = \varepsilon.
\]

Moreover, \( E = \bigcap_{k=1}^{\infty} \{ |f_n - f| < \frac{1}{2^n} \} \).

Thus, for any \( x \in X \), \( x \in E_k \) such that if \( m \geq n_k \) then

\[
E^c = \{ |f_n - f| \geq \frac{1}{2^n} \}, \quad i.e., \quad |f_n(x) - f(x)| < \frac{1}{2^n} \quad \forall x \in E^c.
\]

Hence \( f_n \to f \) uniformly on \( E^c \).
5. Since \( f = \text{Re}f + i\text{Im}f \), we can just consider the case that \( f \) is real-valued. Further, if \( f \) is real-valued, \( f = f^+ - f^- \). So, we can assume that \( f \) is non-negative.

Suppose \( f \) is a simple function, \( f = \sum_{j=1}^{n} a_j \chi_{E_j} \), where \( a_j \in \mathbb{R} \), distinct, \( E_j \subseteq \mathbb{L} \) (class of Lebesgue measurable subsets of \([a, b])\), pairwise disjoint, and \( \bigcup_{j=1}^{n} E_j = [a, b] \).

Let \( \varepsilon > 0 \). By the inner regularity of the Lebesgue measure \( m \), for each \( j \) there exists a compact set \( K_j \subseteq E_j \) such that \( m(K_j) > m(E_j) - \frac{\varepsilon}{n} \).

Let \( E = \bigcup_{j=1}^{n} K_j \). Then \( E \) is compact. Moreover, \( f \mid E \) is continuous, since \( f \) is constant on each \( E_j \) and hence each \( K_j \), and \( \text{dist}(K_i, K_j) > 0 \) if \( i \neq j \) (since \( K_i \subseteq E_i \), \( K_j \subseteq E_j \), \( E_i \cap E_j = \emptyset \), \( i \neq j \), and \( K_i, K_j \) are compact.). Finally,

\[
m(E^c) = m([a, b] \setminus E^c) = m\left(\bigcup_{j=1}^{n} (E_j \setminus K_j)\right) \\
= m\left(\bigcup_{j=1}^{n} (E_j \setminus K_j)^c\right) = m\left(\bigcup_{j=1}^{n} (E_j \setminus E)^c\right) \\
= m\left(\bigcup_{j=1}^{n} (E_j \setminus K_j)^c\right) = \sum_{j=1}^{n} m(E_j \setminus K_j) < \varepsilon.
\]

Now, suppose \( f: [a, b] \to [0, \infty) \) is measurable. There exist simple functions \( \phi_n (n \geq 1, 2, \ldots) \) such that \( 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots \) and \( \phi_n \to f \) pointwise on \([a, b] \). For \( \varepsilon > 0 \), by the previously proved result, for each \( \phi_n \), \exists \text{ compact } K_n \subseteq [a, b] \) such that
\[ m((a, b) \setminus K_n) < \frac{\varepsilon}{2^{n+1}} \quad \text{and} \quad \phi_n|_{K_n^c} \text{ is continuous.} \]

Let \( E_0 = \bigcap_{n=1}^\infty K_n \). \( E_0 \) is compact, and
\[
m((a, b) \setminus E_0) = m((a, b) \setminus \bigcap_{n=2}^\infty K_n^c) = m(\bigcap_{n=1}^\infty (a, b) \setminus K_n^c) \leq \sum_{n=2}^\infty m((a, b) \setminus K_n) < \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}.
\]

Now, \( \phi_n \to f \) on \( E_0 \) pointwise. By Egoroff's Theorem, there exists a measurable subset \( E \subseteq E_0 \) with
\[
m(E_0 \setminus E) < \frac{\varepsilon}{2} \quad \text{and} \quad \phi_n \to f \quad \text{uniformly on} \ E. \]
By the inner regularity, we may assume the \( E \) is compact. Then,
\[
m(E^c) = m((a, b) \setminus E) + m(E_0 \setminus E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Moreover, \( f|_E \) is continuous since \( f|_E \) is the uniform limit of a sequence of continuous functions \( \phi_n \).

6. (i) Since \( f : X \to \mathbb{C} \) and \( g : Y \to \mathbb{C} \) are measurable, there exist simple functions \( \phi_n : X \to \mathbb{C} \) and \( \psi_n : Y \to \mathbb{C} \) such that \( 0 \leq \phi_n \leq \chi_{E_1} \leq \cdots \leq \chi_{E_n} \).
\( \phi_n \to f \) pointwise on \( X \), and \( 0 \leq \psi_n \leq \chi_{F_1} \leq \cdots \leq \chi_{F_n} \) and \( \psi_n \to g \) pointwise on \( Y \). Therefore, \( \phi_n(x) \psi_n(y) \to k(x, y) = f(x) g(y) \) \( \forall x \in X \ \forall y \in Y \). Fix \( n \in \mathbb{N} \). Write \( \phi = \phi_n \) and \( \psi = \psi_n \). We have
\[
\phi(x) = \sum_{j=1}^m a_j \chi_{E_j}(x), \quad a_j \in \mathbb{C}, \ E_j \subseteq X. \quad (\text{disjoint}).
\]
\[
\psi(y) = \sum_{k=1}^l b_k \chi_{F_k}(y), \quad b_k \in \mathbb{C}, \ F_k \subseteq Y. \quad (\text{disjoint}).
\]
Thus, $\phi(x) \psi(y) = \frac{\sum_{l=1}^{m} x_l y_l}{\sum_{l=1}^{m} a_l b_l} \mathbb{1}_{\mathcal{F}}(x) \mathbb{1}_{\mathcal{K}}(y)$

This is a simple function on $\mathbb{R} \times \mathbb{R}$ with respect to $\mu \times \nu$, and is therefore $\mathcal{F} \otimes \mathcal{K}$-measurable. Since all $\phi_n(x), \psi_n(y)$ are $\mathcal{F} \otimes \mathcal{K}$-measurable, their pointwise limit $f(x, y) = \lim_{n \to \infty} \phi_n(x) \psi_n(y)$ is also $\mathcal{F} \otimes \mathcal{K}$-measurable.

(2) Since $f \in L^1(\mu)$ and $g \in L^1(\nu)$, $\mu \left( |f| = \infty \right) = 0, \nu (|g| = \infty) = 0$

and $\mu, \nu$ are $\sigma$-finite on $\{|f| < \infty\}, \{|g| < \infty\}$, respectively. Hence, $\mu \times \nu$ is $\sigma$-finite on $\{|h| < \infty\} = \{|f| < \infty \} \times \{|g| < \infty\}$. Note that $\{|h| = \infty\} \subseteq \{|f| = \infty \} \cup \{|g| = \infty\} \subseteq \{|f| = \infty \} \times \{|g| = \infty\}$. Thus, $\mu \times \nu (\{|h| = \infty\}) = 0 (0 \cdot \infty = 0)$ Hence, applying Tonelli's Theorem on $\{|h| < \infty\} = \{|f| < \infty \} \times \{|g| < \infty\}$, we have

\[
\int_{\mathbb{R} \times \mathbb{R}} |h| \, d(h \times \nu) = \int_{\{|h| < \infty\}} \left[ \int_{\{|g| < \infty\}} |g(y)| \, d\nu(y) \right] |f(x)| \, d\mu(x)
\]

\[
= \int_{\mathbb{R}} |f| \, d\mu \cdot \int_{\mathbb{R}} |g| \, d\nu = 0. \quad \text{Hence, } f \in L^1(\mu \times \nu)
\]

Now, applying Fubini's Theorem, we get

\[
\int_{\mathbb{R} \times \mathbb{R}} f(x, y) \, d(x \times \nu(y))
\]

\[
= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(x, y) \, d\nu(y) \right] \, d\mu(x)
\]

\[
= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} f(x) g(y) \, d\nu(y) \right] \, d\mu(x)
\]

\[
= \left( \int_{\mathbb{R}} f \, d\mu \right) \left( \int_{\mathbb{R}} g \, d\nu \right). \quad \square
\]
Define $\alpha : X \times [0,\infty) \to \mathbb{R}$, $\beta : X \times [0,\infty) \to [0,\infty) \times [0,\infty)$, and $y : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$\alpha(x,y) = f(x) - y$, $\beta(x,y) = (f(x), y)$, and $y(u,v) = u - v$ respectively. Clearly

$$(y \circ \beta)(x,y) = y(\beta(x,y)) = y(f(x), y) = f(x) - y = \alpha(x,y).$$

So, $\alpha = y \circ \beta$.

Clearly $\gamma$ is a continuous function, so it is measurable. Let $A, B \subseteq [0,\infty)$ be Borel sets.

We have $\beta^{-1}(A \times B) = f^{-1}(A) \times B \subseteq \mathcal{B} \times \mathcal{B}$.

In fact, $(a,b) \in \beta^{-1}(A \times B) \iff \beta(a,b) = (f(a), b) \in A \times B \iff f(a) \in A$ and $b \in B \iff a \in f^{-1}(A)$ and $b \in B \iff (a,b) \in f^{-1}(A) \times B$.

Since $\mathcal{B} \otimes \mathcal{B}$, on $[0,\infty) \times [0,\infty)$ is generated by $\{A \times B : A \subseteq [0,\infty), B \subseteq [0,\infty), A, B \in \mathcal{B}\}$, $\beta$ is $\mathcal{B} \otimes \mathcal{B}$-measurable. Thus, $\alpha = y \circ \beta$ is $\mathcal{B} \otimes \mathcal{B}$-measurable. Since $G_f = \{\alpha \geq 0\}$,

$G_f$ is $\mathcal{B} \otimes \mathcal{B}$-measurable.

To prove the identity $\int X \mu \circ \alpha = \int X \mu$, we proceed in several steps.

Step 1: we assume $f$ is bounded. Step 1.1: $M(x) < \infty$. Step 1.2: $M(x)$ is $\sigma$-finite.

Step 2: consider a general $f : X \to [0,\infty)$.

Assume $f$ is bounded, i.e., $\exists M > 0$ such that $0 < f < M$ for all $x \in X$. Assume $M(x) < \infty$. Let $(f_n)$ be a sequence of simple functions on $X$ such that $0 \leq f_n \leq f$ on $X$. 


Then \( X \times [0,M] \setminus G_f = \{ (x,y) \in X \times [0,M] : y > f(x) \} \)
\[ = \bigcap_{n=1}^{\infty} \{ (x,y) \in X \times [0,M] : y > \phi_n(x) \}. \]

Since \( G_f \subseteq X \times [0,M] \), we have
\[
\mu(X) - (\mu \times m)(G_f) = \lim_{n \to \infty} (\mu \times m)\left( \{ (x,y) \in X \times [0,M] : y > \phi_n(x) \} \right).
\]

Fix \( n \) and write \( \phi(x) = \phi_n(x) = \sum_{j=1}^{m} a_j \chi_{E_j}(x) \) with \( E_j \subseteq X \) \((1 \leq j \leq m)\), \( \chi = \bigcup_{j=1}^{m} \chi_{E_j} \) (disjoint union), and \( a_j \geq 0 \). Note \( \phi_n \leq f < M \) so, \( a_j < M \). Moreover,
\[
\{ (x,y) \in X \times [0,M] : y > \phi(x) \} = \bigcup_{j=1}^{m} \{ (x,y) \in E_j \times [0,M] : y > a_j \} = \bigcup_{j=1}^{m} (E_j \times (a_j, M]).
\]

This is a disjoint union. Thus,
\[
(\mu \times m)\left( \{ (x,y) \in X \times [0,M] : y > \phi(x) \} \right) = \sum_{j=1}^{m} \mu(E_j)(M-a_j) = M \mu(X) - \sum_{j=1}^{m} a_j \mu(E_j) = M \mu(X) - \int f \, dm.
\]

Consequently, from \((\ast)\), by the Monotone Convergence Thm.
\[ \mu(X) - (\mu \times m)(G_f) = M \mu(X) - \lim_{n \to \infty} \int \phi_n \, dm. \]

Hence \( (\mu \times m)(G_f) = \int f \, dm. \)

If \( \mu \) is \( \sigma \)-finite, then \( \exists X_j \subseteq M, X_j \uparrow \bigcap_{j=1}^{\infty} X_j = X \) and \( \mu(X_j) < \infty \) \((V_j \geq 1)\). Denote
\[ G_f(X_j) = \{ (x,y) \in X_j \times [0,\infty) : y \leq f(x) \}. \]

Then, as shown before, each \( G_f(X_j) \) is measurable.
Since \( x_j \) increases, \( G_f(x_j) \) increases. Since \( \partial x_j = x \),

\( G_f(x_j) = G_f \).

Then

\[
(\chi \times m)(G_f) = \lim_{n \to \infty} \int \chi \times m(G_f(x_j)) \overset{\text{continuity of measure}}{=} \lim_{n \to \infty} \int \chi \times m(G_f) \overset{\text{since } \chi \times m(x_j) < \infty}{=} \lim_{n \to \infty} \int \chi \times m \overset{\text{and the previously proven result}}{=} \int \chi \times m \overset{\text{MCT}}{=}
\]

Now, assume a general case: \( f : X \to [0, \infty) \).

Let \( f_n = \min(f, n) \). Then \( f_n : X \to [0, \infty) \) is measurable.

\( 0 \leq f_1 \leq f_2 \leq \ldots \) \( f_n(x) \to f(x) \) \( \forall x \in X \). Define

\[
G_{f_n} = \{(x, y) \in X \times [0, \infty) : y \leq f_n(x)\} \quad (n = 1, 2, \ldots)
\]

Then \( G_f \subseteq G_{f_n} \forall n \). Clearly \( G_{f_n} \subseteq G_f \).

If \( (x, y) \in G_f \) then \( x \in X, 0 \leq y \leq f(x) \). Let \( n \) be such that \( n > f(x) \) \( \left[ \text{possible since } f(x) < \infty \right] \)

then \( f(x) = f_n(x) = \min(f(x), n) \). Hence, \( (x, y) \in G_{f_n} \).

Thus, \( G_f = \bigcap_{n=1}^{\infty} G_{f_n} \). Consequently, by the continuity of measure, what's proved above, and the MCT,

\[
(\chi \times m)(G_f) = \lim_{n \to \infty} (\chi \times m)(G_{f_n}) = \lim_{n \to \infty} \int \chi \times m \overset{\text{MCT}}{=} \int \chi \times m.
\]

\[\square\]
8. The left-hand side is
\[
\int_0^1 \int_0^\infty (e^{-xy} - 2e^{-2xy}) \, dy \, dx
\]
\[
= \int_0^1 \left( \int_0^\infty e^{-xy} \, dy \right) \frac{1}{x} (e^{2x} - e^x) \, dx
\]
\[
= \int_0^1 0 \, dx = 0.
\]

The right-hand side is
\[
\int_0^\infty \int_0^1 (e^{-xy} - 2e^{-2xy}) \, dx \, dy
\]
\[
= \int_0^\infty \left( \int_0^1 \frac{2x}{y} e^{-xy} \, dx \right) (e^{y} - e^{2y}) \, dy
\]
\[
= \int_0^\infty \frac{1}{y} (e^{-y} - e^{-2y}) \, dy
\]
\[
< 0 \quad \text{since} \quad e^{-y} - e^{-2y} < 0 \quad \text{for} \quad y > 0. \quad \square
\]

9. Let \( f(x, r) = \sin x \, e^{-x^2} \) \((x \geq 0, r > 0)\). Let \( A > 0 \).

We have
\[
\int_0^A \int_0^\infty |f(x, r)| \, dx \, dr = \int_0^A |\sin x| \left( \int_0^\infty e^{-x^2} \, dx \right) \, dx
\]
\[
= \int_0^A |\sin x| \left( -\frac{1}{2} e^{-x^2} \right)_{x=0}^{x=\infty} \, dx
\]
\[
= \int_0^A \frac{|\sin x|}{x} \, dx < \infty.
\]

Since \( \frac{\sin x}{x} \to 1 \) as \( x \to 0 \). Thus, by Tonelli's Theorem,
\[
f(C L^1(0, A) \times (0, \infty)) \quad \text{Hence, using the fact that}
\]
\[
V_k = \int_0^\infty e^{-x^2} \, dx \quad (A \times \infty), \quad \text{we have by Fubini's}
\]
\[
\int_0^A \sin x \, dx = \int_0^A \int_0^\infty \sin x \, e^{-x^2} \, dt \, dx
\]
\[
= \int_0^\infty \int_0^A \sin x \, e^{-x^2} \, dx \, dt.
\]
By the integration by parts, we get
\[
\int_0^A \sin x e^{-xt} \, dx = \int_0^A (-\cos x) e^{-xt} \, dx
\]
\[
= -\cos x e^{-xt} \bigg|_{x=0}^{x=A} - \int_0^A (\cos x) (-tf) e^{-xt} \, dx
\]
\[
= -(\cos A) e^{-At} + t \int_0^A \cos x e^{-xt} \, dx
\]
\[
= -(\cos A) e^{-At} + t \left[ \sin x e^{-xt} \bigg|_{x=0}^{x=A} - \int_0^A \sin x (-t) e^{-xt} \, dx \right]
\]
\[
= -(\cos A) e^{-At} + t \left[ \sin x e^{-xt} \bigg|_{x=0}^{x=A} - \int_0^A \sin x (-t) e^{-xt} \, dx \right]
\]
Thus,
\[
\int_0^A \sin x e^{-xt} \, dx = \frac{1}{1+it} \left[ -e^{-At} \cos A + 1 - t e^{-At} \sin A \right]
\]
This is controlled by \(\frac{1}{1+it}\) for large A. So, by the dominant convergence theorem, we have
\[
\int_0^A \frac{\sin x}{x} \, dx = \int_0^\infty \frac{1}{1+it} \left[ -e^{-At} \cos A + 1 - t e^{-At} \sin A \right] \, dt
\]
\[
\rightarrow \arctan(0) - \int_0^\infty \frac{0}{t} \, dt = \frac{\pi}{2}.
\]

10. Let \(H(x,t) = X(x,a) (t) \frac{f(t)}{t} \) (\(0 < x, t < a\)).

\[
X(x,a) (t) = X_{(0,t)} (x) = \begin{cases} 1 & \text{if } x < t \\ 0 & \text{if } x \geq t \end{cases}
\]
Thus,
\[
\int_0^a \int_0^a |H(x,t)| \, dx \, dt
\]
\[
= \int_0^a \frac{|f(t)|}{t} \left[ \int_0^a X_{(0,t)} (x) \, dx \right] \, dt
\]
\[
= \int_0^a \frac{|f(t)|}{t} \, dt
\]
\[
= \int_0^a |f(t)| \, dt < \infty
\]
Since \(f \in L^1([0,1])\). Thus, \(H \in L^1([0,a] \times [0,a])\).
Now, by Fubini’s Theorem,
\[ \int_{(0,a)} g \, dm = \int_0^a \int_0^a \frac{f(t)}{t} \, dt \, dx = \int_0^a \int_0^a \frac{f(t)}{t} \, dx \, dt \]
\[ = \int_0^a \int_0^a K(x,a)(t) \frac{f(t)}{t} \, dx \, dt \]
\[ = \int_0^a \int_0^a H(t) \, dt \, dx \]
\[ = \int_0^a \int_0^a \frac{f(t)}{t} \left[ \int_0^a K(x,a)(t) \, dx \right] \, dt \]
\[ = \int_0^a \int_0^a \frac{f(t)}{t} \left[ \int_0^a X(x,a,t) \, dx \right] \, dt \]
\[ = \int_0^a f(t) \, dt \int_0^a \frac{f(t)}{t} \, dx \]
\[ = \int_0^a f(t) \, dt = \int_{(0,a)} f \, dm. \]