1. (1) Suppose $E$ is $\nu$-null. Let $X = P \cup N$ be a Hahn decomposition for $\nu$, where $P, N \in \mathcal{M}$ are disjoint, $P$ is $\nu$-positive, and $N$ $\nu$-negative. Then, $\nu^+(A) = \nu(A \cap P)$ and $\nu^-(A) = -\nu(A \cap N)$ for any $A \in \mathcal{M}$, and $\nu = \nu^+ - \nu^-$ with $\nu^+ \nu^-$ is the Jordan decomposition of $\nu$. Since $E$ is $\nu$-null, $\nu(E \cap P) = 0$, $\nu(E \cap N) = 0$. Thus, $\nu(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) = 0$.

Suppose now $\nu(E) = 0$. Then, $\nu^+(E) = 0$, $\nu^-(E) = 0$. Thus, for any $F \in \mathcal{M}$, $F \subseteq E$, $\nu^+(F) \leq \nu^+(E) = 0$. So, $\nu^+(F) = 0$. Similarly, $\nu^-(F) = 0$. Thus, $\nu(F) = \nu^+(F) - \nu^-(F) = 0$. Hence, $E$ is $\nu$-null.

(2) Suppose $\nu \ll \mu$. Then $E, F \in \mathcal{M}$, $X = E \cup F$, $E \cap F = \emptyset$. $E$ is $\nu$-null and $F$ is $\mu$-null. By (1), $\nu(E) = 0$ and $\mu(F) = 0$. Hence, $\nu^+(E) \leq \nu(E) = 0$ i.e., $\nu^+(E) = 0$. Similarly, $\nu^-(E) = 0$. Since $\nu^+, \nu^-$ are positive measures, $E$ is $\nu^+$-null and $\nu^-$-null. Thus, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Conversely, assume $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Then, there exist $E^+, E^-, F^+, F^- \in \mathcal{M}$, $X = E^+ \cup F^+$, $E^- \cap F^- = \emptyset$. $E^+$ is $\nu^+$-null i.e., $\nu^+(E^+) = 0$, $F^+$ is $\mu$-null, i.e., by (1), $\mu(F^+) = 0$. Similarly, $X = E^- \cup F^-$, $E^+ \cap F^- = \emptyset$, $E^- \cap F^+ = \emptyset$, $\nu^-(E^-) = 0$ and $\mu(F^-) = 0$. Let $F = F^+ \cup F^- \in \mathcal{M}$ and $E = E^+ \cap F^+ \cap F^- \cap F^-$.

Since $E^+ \cap F^+ = \emptyset$ and $E^- \cap F^- = \emptyset$, $E \cap F = (E^+ \cap F^-) \cap (F^+ \cup F^-)$.
\[(E^+ \cup E^-) \cup (E^+ \cup E^-) = X\] 
Since \[E^+ = E^-,\ E \cup E^+ = X\]
\[E U F = (E^+ \cup E^-) \cup (F^+ \cup F^-) = (F^+ \cup F^-) \cup (E^+ \cup F^-) = (F^+ \cup F^-) \cup (E^+ \cup F^-) = X.\]
Moreover, \[|\mu_1(F)| = |\mu_1(F^+) + |\mu_1(F^-)| = 0.\] So, \(F\) is \(\mu\)-null. \[|\mu_1(E)| = \mu^+(E) + \mu^-(E) \leq \mu^+(E) + \mu^-(E) = 0.\] So \(E\) is \(\nu\)-null. Thus, \(\mu \perp \nu\).

2. (1) Let \(X = P \cup N\) be a Hahn decomposition for \(\nu\). \(\phi\) is \(\nu\)-positive, \(N\) \(\nu\)-negative, \(P \cap N = \emptyset\). Then \(\nu^+(E) = \nu(E^+)\) and \(\nu^-(E) = -\nu(E^-)\). Since \(E \cap P \in \mathcal{M}\) and \(E \cap N \in \mathcal{E}\),
\[\nu^+(E) = \sup \{ \nu(F) : F \in \mathcal{M}, \ F \leq E \}.\]
On the other hand, if \(F \in \mathcal{M}, F \subseteq E\), then
\[\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) = \nu^+(F \cap P) \leq \nu^+(F) \leq \nu^+(E)\]
Hence, \(\nu^+(E) = \sup \{ \nu(F) : F \in \mathcal{M}, F \subseteq E \} \).
Similarly, \(\nu^-(E) = -\nu(E \cap N) = -\inf \{ \nu(F) : F \in \mathcal{M}, F \subseteq E \} \).
If \(F \in \mathcal{M}\) and \(F \subseteq E\), then
\[\nu(F) = \nu^+(F) - \nu^-(F) \geq -\nu(F) \geq -\nu(E).\]
Hence
\[\nu^-(E) = -\inf \{ \nu(F) : F \in \mathcal{M}, F \subseteq E \} \]. Finally,
\[\nu^{-}(E) = \inf \{ \nu(F) : F \in \mathcal{M}, F \subseteq E \}.\]

(2) If \(E = \bigcup_{j=1}^{n} E_j\) with \(E_j \in \mathcal{M}\) \((j = 1, \ldots, n)\) disjoint, then
\[\sum_{j=1}^{n} |\mu_1(E_j)| \leq \sum_{j=1}^{n} |\mu_1(E_j)| = |\mu_1(E)| = |\mu_1(E_j)| = |\mu_1(E_j)|.\]
Hence, \(|\mu_1(E)| = \sup \{ \sum_{j=1}^{n} |\mu_1(E_j)| : n \in \mathbb{N}, E_j \cap E_k = \emptyset, E_j \in \mathcal{M}, E_k \in \mathcal{M}, \j)\text{-disjoint} \}
\[E = \bigcup_{j=1}^{n} E_j.\]
Now, let \(X = P \cup N\) be a Hahn decomposition for \(\nu\). \(P \in \mathcal{M}\), \(P \cap N = \emptyset\). \(P\), \(N\) \(\nu\)-positive, \(\nu\)-negative, respectively. Set \(E_j = P \cap E_j, E_j = N \cap E_j\). Then,
3. (1) Let \( V = V^+ - V^- \) with \( V^+ \cup V^- \) be the Jordan decomposition for \( V \). Then \( L^+(V) = L^+(V^+) \cap L^-(V^-) \).

If \( f \in L^+(V) \), then \( f \in L^+(V^+) \cap L^-(V^-) \). Hence
\[
\int |f| \, dv = \int |f| \, dv^+ + \int |f| \, dv^- < \infty, \quad \text{i.e., } f \in L^1(V).
\]

Conversely, if \( f \in L^1(V) \Rightarrow \int |f| \, dv < \infty \). Thus
\[
\int |f| \, dv = \int |f| \, dv^+ + \int |f| \, dv^- < \infty \quad \text{and} \quad f \in L^1(V^+) \cap L^1(V^-)
\]

= \( L^1(V) \).

(2) By (1), \( f \in L^1(V^+) \cap L^1(V^-) \Leftrightarrow f \in L^1(V) \)
\[
\int |f| \, dv = \int \left| f \right| \, dv^+ - \int \left| f \right| \, dv^- \leq \int |f| \, dv^+ + \int |f| \, dv^- = \int |f| \, dv.
\]

(3) Let \( E \in \mathcal{M} \). If \( f : X \to C_0(\mathbb{R}) \) is measurable and \( |f| \leq 1 \) then \( \left\| f \right\|_1 \leq \int |f| \, dv \leq \int |f| \, dv + \int |f| \, dv^- = 1 \int 1 \, dv = 1 \int 1 \, dv \).

Let \( X = V \cup N \) be a Hahn decomposition with \( P, N \) positive, negative sets for \( V \), disjoint. Let \( f = X_p - X_N \), then \( |f| = 1 \), and
\[
\int f \, dv = \int_X f \, dv = \int_X \left( X_p - X_N \right) \, dv = V(E \cap P) - V(E \cap N) = V^+(E) + V^-(E) = |V| \, (E).
\]

Hence \( |V| \, (E) = \sup \left\{ \left\| f \right\|_1 : |f| \leq 1 \right\} \).
4. (1) \( \forall E \in \mathcal{P} : \, |u(E)| \leq \int |f| \, du \leq \int |f| \, du < \infty \), since \( f \) is integrable.

So, \( u(E) \neq \pm \infty \). Clearly \( u(\emptyset) = 0 \). Let \( E_j \in \mathcal{P} \) \((j = 1, 2, \ldots)\) be disjoint, and let \( E = \bigcup_{j=1}^{\infty} E_j \). Then,

\[
\sum_{j=1}^{\infty} \int_{x \in E_j} |f| \, dx = \int_{x \in E} \sum_{j=1}^{\infty} |f| \, dx = \int_{x \in E} f \, dx \leq \int_{x \in E} f \, dx < \infty.
\]

Consequently,

\[
u(E) = \int_{x \in E} f \, dx = \sum_{j=1}^{\infty} \int_{x \in E_j} f \, dx = \sum_{j=1}^{\infty} \nu(E_j),
\]

Hence \( \nu \) is a signed measure.

(2) \( \mathcal{P} = \{ f > 0 \} \), \( \mathcal{N} = \{ f < 0 \} \), \( \mathcal{X} = \mathcal{P} \cup \mathcal{N} \) is a Hahn decomposition for \( \nu \), as \( \mathcal{P}, \mathcal{N} \) disjoint, and they are positive, negative for \( \nu \), respectively.

\[
u^+(E) = \nu(E \cap \mathcal{P}) = \int_{x \in E} f \, dx = \int_{x \in E} |f| \, dx \geq 0,
\]

\[
u^-(E) = -\nu(E \cap \mathcal{N}) = -\int_{x \in E} f \, dx = \int_{x \in E} |f| \, dx \geq 0.
\]

\[
u(E) = \nu^+(E) + \nu^-(E).\]

So, \( \nu^+(E) + \nu^-(E) = \|\nu\| \).

5. (1) \( \Rightarrow \) (2) Let \( E \in \mathcal{M} \). Suppose \( \nu \ll \mu \). If \( \mu(E) = 0 \) then for a Hahn decomposition \( \mathcal{X} = \mathcal{P} \cup \mathcal{N} \) \((\text{disjoint})\) for \( \nu \) \((\mathcal{P}, \mathcal{N} \) positive, negative for \( \nu \), respectively) we have \( \mu(E \cap \mathcal{P}) = 0 \), \( \mu(E \cap \mathcal{N}) = 0 \). But \( \nu \ll \mu \).

So, \( \nu^+(E) = \nu(E \cap \mathcal{P}) = 0 \), \( \nu^- (E) = \nu(E \cap \mathcal{N}) = 0 \). Hence

\[
u(E) = \nu^+(E) + \nu^-(E) = 0. \]

i.e., \( |\nu| < \mu \).
(2) \Rightarrow (3). If E \in M_1 and \mu(E) = 0, then |\nu(E)| = 0 by (1). Hence \nu^+(E) = \nu^-(E) = |\nu(E)| = 0. Similarly, \nu^-(E) = 0. So, \nu^+=\nu and \nu^-=\nu.

(3) \Rightarrow (1). Let E \in M_1 and \mu(E) = 0. Then, by (3), 
\nu^+(E) = \nu^-(E) = 0. Hence \|\nu(E)\| = |\nu^+(E) - \nu^-(E)| 
\leq |\nu^+(E)| + |\nu^-(E)| = 0. i.e., \nu \ll \mu.

6. Let \epsilon > 0. Since \lim f_n \to f \in L_1(\mu), there exists N \in \mathbb{N} such that \|f_n - f\|_1 < \epsilon/2 \quad \forall n > N. Since all f, f_1, ..., f_N \in L_1(\mu), there exists \delta > 0 such that for any E \in \mathcal{M} with \mu(E) < \delta,
\|f_n\|_1 < \frac{\epsilon}{2}, \quad \|f\|_1 < \epsilon \quad (s=1, ..., N).

Therefore, for any n \in \mathbb{N} since \|f_n\|_1 
\leq \|f_n - f\|_1 + \|f\|_1 we have
\|f_n\|_1 \leq \max \{\|f\|_1, \|f_n - f\|_1\} 
\leq \|f\|_1 + \|f_n - f\|_1 < \delta.

Hence \{f_n\}_{n=1} is uniformly integrable.

7. (1) If E \in \mathcal{M} and \mu(E) = 0, then E = \emptyset. Hence m(\emptyset) = 0. i.e., m \ll \mu.

Suppose f \in L_1(\mu) such that dm = f d\mu. Let \chi \in \chi = [0,1] and E = \{x \in \chi \mid f(x) = 1\}. Then,
o = m(E) = \int_{E_1} f \, d\mu = \int_{\chi} f(x) \mu(\{x\}) = f(x).
So \mu(E) = 0. But then 1 = m(\chi) = \int_{\chi} f \, d\mu = \int_{\chi} 1 \, d\mu = \mu(\chi), a contradiction. So, no f \in L_1(\mu) will satisfy dm = f d\mu.
(2) Suppose $\mu = \lambda + \nu$ for some signed measures $\lambda$ and $\nu$ on $X$, with $\lambda \ll \mu$ and $\nu \ll \mu$. Since $\lambda \ll \mu$, there exist $A, B \in \mathcal{B}(\mathbb{R})$, disjoint such that $X = A \cup B$, $A$ is null for $\lambda$ and $\mu(B) = 0$. We have $A \neq \emptyset$, for otherwise, $A = \emptyset$, $1 = \lambda(X) = \lambda(A) \ll \lambda(B) = 0$. This is impossible. Now, let $y \in \lambda(A)$. Then $m(E) = 0$. Since $\mu \ll \lambda$, $E \in \mathcal{B}(\mathbb{R})$. Thus, $1 = \lambda(E) = \lambda(E) + \nu(E) = 10^2$, but $A$ is $\lambda$-null, so $\lambda(E) = 0$, a contradiction. So, $\mu$ permits no Lebesgue decomposition with respect to $\mu$.

8. Since $\lambda = \mu + \nu$ and $\nu \ll \mu$, we have that $\mu \ll \nu$ and that $d\lambda = d\mu + d\nu$. Since $\mu \ll \lambda$ and $f = \frac{d\nu}{d\lambda}$, we have 

$$1 = \frac{d\lambda}{d\lambda} = \frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda} = \frac{d\mu}{d\lambda} + f$$

Thus $1 - f = \frac{d\mu}{d\lambda}$. We show that $f < 1$ $\mu$-a.e., or equivalently, $\lambda$-a.e. If there existed $E \in \mathcal{M}$ with $\mu(E) > 0$ (equivalently $\lambda(E) > 0$) such that $f > 1$ on $E$. Then, by the fact that $1 - f = \frac{d\mu}{d\lambda}$,

$$\int_E (1 - f) \, d\lambda = \int_E \frac{d\mu}{d\lambda} \, d\lambda = \int_E d\mu = \mu(E) > 0.$$  

But, $\int_E (1 - f) \, d\lambda \leq 0$ since $1 - f \leq 0$ on $E$. This is a contradiction. Thus, $f < 1$ $\mu$-a.e. Hence

$$1 - f = \frac{d\mu}{d\lambda}, \text{ $\mu$-a.e. (or equivalently, $\lambda$-a.e.)}$$

But $f = \frac{d\mu}{d\lambda} \geq 0$ since $\mu \ll \lambda$. Thus, $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} = f / \frac{d\mu}{d\lambda} = f$. 


9. Note that $\nu = \mu|_{\mathcal{B}}$ is a finite measure on $(X, \mathcal{B})$. Define $\lambda : \mathcal{B} \to \mathbb{C}$ by $\lambda(E) = \frac{1}{\nu} \int_E f \, d\nu$ for any $E \in \mathcal{B}$. Since $f \in L^1(\nu)$ and $\nu = \mu|_{\mathcal{B}}$, if $E \in \mathcal{B}$ and $\nu(E) = 0$, then $\lambda(E) = 0$. Thus $\nu << \lambda$.

By the Lebesgue–Radon–Nikodym Theorem, there exists $g \in L^1(\nu)$ such that $d\nu = gd\nu$ i.e., $\lambda(E) = \int_E g \, d\nu$, hence $\int f \, d\nu = \int g \, d\nu$ for all $E \in \mathcal{B}$. In particular, $g$ is $\mathcal{B}$-measurable.

Suppose $h \in L^1(\nu)$ and $\int g \, d\nu = \int h \, d\nu$ for all $E \in \mathcal{B}$. Then, setting $u = g - h \in L^1(\nu)$, we have $\int u \, d\nu = 0$ for all $E \in \mathcal{B}$. Hence $u = 0$, i.e., $g = h$, $\nu$-a.e.

10. If $f \in L^1(\mathcal{B})$ such that $|f| = 1$ $\nu$-a.e. and $d\nu = f \, d\mathcal{B}$. Since $\nu(X) = \nu(X) = \int f \, d\nu = \int \text{Re} f \, d\nu$,

\[ \int \text{Im} f \, d\nu = \int \text{Im} f \, d\nu = \int |f| \, d\nu = \int |f| \, d\nu. \]

Since $\nu(X)$ is a real number, $\int \text{Im} f \, d\nu = 0$, and $\int \text{Re} f \, d\nu = |\nu(X)| = \int |f| \, d\nu$.

Consequently, $\int (1 - \text{Re} f) \, d\nu = 0$. But $|f| = 1$, $\nu$-a.e. So, $\text{Re} f \leq 1$ $\nu$-a.e. and $|1 - \text{Re} f| \leq 1$ $\nu$-a.e.

Thus, by $\int (1 - \text{Re} f) \, d\nu = 0$, we have $|1 - \text{Re} f| = 0$ $\nu$-a.e. Since $|f| = 1$ $\nu$-a.e., $|f| = \sqrt{\text{Re} f^2 + \text{Im} f^2}$, we have $\text{Re} f = 0$ $\nu$-a.e. and $f = \text{Re} f = 1$ $\nu$-a.e.

Finally, $d\nu = f \, d\nu = d\nu$ and $\nu = \nu$. 