1. Prove the following:
   (1) In a topological space, the union of two compact subsets is compact;
   (2) In a Hausdorff topological space, the intersection of two compact subsets is compact.

2. Let $X$ and $Y$ be two topological spaces. Assume that $Y$ is compact. Prove that the coordinate map $\pi_1 : X \times Y \to X$ is a closed map, i.e., $\pi_1(E)$ is a closed subset of $X$ if $E$ is a closed subset of $X \times Y$. (You can assume that $X$ and $Y$ are metric spaces with $Y$ compact, if you feel this will help.)

3. Prove that a sequentially compact topological space is countably compact.

4. Let $X$ be a compact topological space and $f \in C(X, \mathbb{R})$. Prove that there exist $a, b \in X$ such that $f(a) = \min_{x \in X} f(x)$ and $f(b) = \max_{x \in X} f(x)$.

5. Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the $n$-sphere $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

6. Prove that every open set in a second countable locally compact Hausdorff space is $\sigma$-compact.

7. (Optional; cf. Theorem 0.25 on page 15.) Let $X$ be a metric space. Prove that the following three statements are equivalent:
   (1) $X$ is compact;
   (2) $X$ is sequentially compact;
   (3) $X$ is complete and totally bounded (i.e., for any $\varepsilon > 0$, there are finitely many balls of radius $\varepsilon$ such that their union is $X$.