Math 240B: Real Analysis, Winter 2016

Homework Assignment 5

Due Friday, February 12, 2016

1. Let X be a finite-dimensional normed vector space. Prove the following:
   (1) X is complete;
   (2) Any two norms on X are equivalent.

   (See Exercise 6 on page 155.)

2. Let $\alpha \in (0,1]$. Denote by $C^{0,\alpha}([0,1])$ the vector space of all real-valued, Hölder continuous functions on $[0,1]$ of exponent α : $f \in C^{0,\alpha}([0,1])$ means that $f \in C([0,1])$ and
$$[f]_\alpha := \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty.$$
   Prove that
$$\|f\| = \sup_{x\in[0,1]} |f(x)| + [f]_\alpha$$
   is a norm on $C^{0,\alpha}([0,1])$, and with this norm, $C^{0,\alpha}([0,1])$ is a Banach space.

3. Let X be a normed vector space and M a closed subspace of X. Prove the following:
   (1) $\|x+M\| = \inf\{\|x+y\| : y \in M\}$ is a norm on the quotient space X/M;
   (2) For any $\varepsilon > 0$ there exists $x \in X$ such that $\|x\| = 1$ and $\|x+M\| \geq 1-\varepsilon$;
   (3) The projection map $\pi(x) = x + M$ from X to X/M has norm 1;
   (4) If X is complete, so is X/M.

4. Let X be an infinite-dimensional normed vector space. Prove the following:
   (1) There is a sequence $\{x_j\}$ in X such that $\|x_j\| = 1$ for all j and $\|x_j - x_k\| \geq 1/2$ for $j \neq k$.
   (2) X is not locally compact.

   (See Exercise 19 on page 160.)

5. If X is a Banach space and X^* is separable, then X is separable. (See Exercise 25 on page 160.)