Math 240B: Real Analysis, Winter 2020

Homework Assignment 3
Due Wednesday, January 29, 2020

Unless otherwise stated, $(X, \mathcal{M}, \mu)$ denotes a measure space.

1. Let $1 \leq p < r \leq \infty$. Prove the following:
   (1) The space $L^p(\mu) \cap L^r(\mu)$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$. Moreover, if $p < q < r$ then the inclusion map $L^p \cap L^r \rightarrow L^q$ is continuous;
   (2) The space $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h\}$.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $1 < p < \infty$, and $f_n, f \in L^p(\mu)$ $(n = 1, 2, \ldots)$. Assume $\sup_{n \geq 1} \|f\|_p < \infty$ and $\lim_{n \rightarrow \infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ for any $E \in \mathcal{M}$. Prove that $f_n \rightarrow f$ weakly in $L^p(\mu)$. (It turns out that these conditions are also necessary for $f_n$ to converge to $f$ weakly in $L^p(\mu)$. The first condition follows from the fact that the space $L^p(\mu)$ is the dual of $L^q(\mu)$ with $q$ the conjugate of $p$ and the Principle of Uniform Boundedness for bounded linear operators; see Theorem 5.13.)

3. Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_X$ (the Borel $\sigma$-algebra of $X$), and $\mu = m$ (the Lebesgue measure).
   (1) Construct $f, f_n \in L^2(X)$ $(n = 1, 2, \ldots)$ such that $f_n \rightarrow f$ weakly in $L^2(X)$ but $f_n \not\rightarrow f$ in measure and $f_n \not\rightarrow f$ a.e.
   (2) Construct $f, f_n \in L^2(X)$ $(n = 1, 2, \ldots)$ such that $f_n \rightarrow f$ in measure and a.e. but $f_n \not\rightarrow f$ weakly.
   (You can use the sufficient and necessary conditions for the weak convergence in $L^2$ stated in Problem 2.)

4. (Hilbert’s Inequality) Let $X = (0, \infty)$, $\mathcal{M} = \mathcal{B}_X$ (the Borel $\sigma$-algebra on $X$), and $\mu = m$ (the Lebesgue measure). Let $1 < p < \infty$. For any $f \in L^p(0, \infty)$, define $Tf$ by $Tf(x) = \int_0^\infty \frac{f(y) \, dy}{x + y}$ for any $x \in (0, \infty)$. Also define $C_p = \int_0^\infty \frac{dx}{x^{1/p}(x + 1)}$. Prove that $\|Tf\|_p \leq C_p \|f\|_p$.

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two $\sigma$-finite measure spaces and $K \in L^2(\mu \times \nu)$. Prove the following:
   (1) If $f \in L^2(\nu)$, then $\int_Y |K(x, y)f(y)| \, d\nu < \infty$ for $\mu$-a.e. $x \in X$;
   (2) If $f \in L^2(\nu)$ and $Tf(x) = \int_Y K(x, y)f(y) \, d\nu$, then $Tf \in L^2(\mu)$ and $\|Tf\|_2 \leq \|K\|_2 \|f\|_2$.

6. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Define $Tf(x) = x^{-1/p} \int_0^x f(t) \, dt$. Prove that $T$ is a bounded linear map from $L^q((0, \infty))$ to $C_0((0, \infty))$. 