Math 240B: Real Analysis, Winter 2020

Homework Assignment 3

Due Wednesday, January 29, 2020

Unless otherwise stated, $(X, \mathcal{M}, \mu)$ denotes a measure space.

1. Let $1 \le p < r \le \infty$. Prove the following:
   (1) The space $L^p(\mu) \cap L^r(\mu)$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$. Moreover, if $p < q < r$ then the inclusion map $L^p(\mu) \cap L^r(\mu) \to L^q(\mu)$ is continuous;
   (2) The space $L^p(\mu) + L^r(\mu)$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h$ with $g \in L^p(\mu)$ and $h \in L^r(\mu)\}$.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $1 < p < \infty$, and f_n, $f \in L^p(\mu)$ $(n = 1, 2, \dots)$. Assume $\sup_{n \ge 1} \|f\|_p < \infty$ and $\lim\limits_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$ for any $E \in \mathcal{M}$. Prove that $f_n \to f$ weakly in $L^p(\mu)$. (It turns out that these conditions are also necessary for f_n to converge to f weakly in $L^p(\mu)$. The first condition follows from the fact that the space $L^p(\mu)$ is the dual of $L^q(\mu)$ with q the conjugate of p and the Principle of Uniform Boundedness for bounded linear operators; see Theorem 5.13.)

3. Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_X$ (the Borel σ-algebra of X), and $\mu = m$ (the Lebesgue measure).
   (1) Construct f, $f_n \in L^2(X)$ $(n = 1, 2, \dots)$ such that $f_n \to f$ weakly in $L^2(X)$ but $f_n \not\to f$ in measure and $f_n \not\to f$ a.e.
   (2) Construct f, $f_n \in L^2(X)$ $(n = 1, 2, \dots)$ such that $f_n \to f$ in measure and a.e. but $f_n \not\to f$ weakly.

   (You can use the sufficient and necessary conditions for the weak convergence in L^2 stated in Problem 2.)

4. (Hilbert's Inequality) Let $X = (0, \infty)$, $\mathcal{M} = \mathcal{B}_X$ (the Borel σ-algebra on X), and $\mu = m$ (the Lebesgue measure). Let $1 < p < \infty$. For any $f \in L^p(0, \infty)$, define Tf by $Tf(x) = \displaystyle\int_0^\infty \frac{f(y)\, dy}{x + y}$ for any $x \in (0, \infty)$. Also define $C_p = \displaystyle\int_0^\infty \frac{dx}{x^{1/p}(x+1)}$. Prove that $\|Tf\|_p \le C_p \|f\|_p$.

5. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two σ-finite measure spaces and $K \in L^2(\mu \times \nu)$. Prove the following:
   (1) If $f \in L^2(\nu)$, then $\displaystyle\int_Y |K(x, y) f(y)| \, d\nu(y) < \infty$ for μ-a.e. $x \in X$;
   (2) If $f \in L^2(\nu)$ and $Tf(x) = \displaystyle\int_Y K(x, y) f(y) \, d\nu(y)$, then $Tf \in L^2(\mu)$ and $\|Tf\|_2 \le \|K\|_2 \|f\|_2$.

6. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Define $Tf(x) = x^{-1/p} \displaystyle\int_0^x f(t)\, dt$. Prove that T is a bounded liner map from $L^q((0, \infty))$ to $C_0((0, \infty))$.