1. Let \( n \in \mathbb{N} \) and \(-\infty < a = x_0 < x_1 < \cdots < x_n = b < \infty\). Denote by \( \mathcal{P}_n \) the set of all real polynomials of degree \( \leq n \).

   (1) Define \( l_j \in \mathcal{P}_n \) by \( l_j(x) = \prod_{i=0, i \neq j}^{n} (x-x_i)/(x_j-x_i) \) (\( j = 0, \ldots, n \)). Show that \( l_j(x_k) = \delta_{jk} \) (\( \delta_{jk} = 1 \) if \( j = k \) and \( 0 \) if \( j \neq k \)).

   (2) Let \( f \in C([a,b]) \) and define \( L_n f \in \mathcal{P}_n \) by \( (L_n f)(x) = \sum_{j=0}^{n} f(x_j) l_j(x) \) (called the Lagrange interpolation of \( f \)). Prove that \( L_n f \) is the unique polynomial in \( \mathcal{P}_n \) that satisfies \( (L_n f)(x_j) = f(x_j) \) (\( j = 0, 1, \ldots, n \)).

   (3) Prove that the operator norm of the linear operator \( L_n : C([a,b]) \to C([a,b]) \) (where \( C([a,b]) \) is equipped with the maximum norm) is given by \( \|L_n\| = \max_{x \in [a,b]} \sum_{j=0}^{n} |l_j(x)| \).

2. Consider the Banach space \( L^\infty([0,1]) \) and its subspace \( C([0,1]) \).

   (1) Let \( x_0 \in (0,1) \) and define \( F : C([0,1]) \to \mathbb{C} \) by \( F(f) = f(x_0) \). Prove that \( F \in C([0,1])^* \).

   (2) By Hahn–Banach Theorem, there exists \( \tilde{F} \in L^\infty([0,1])^* \) such that \( \tilde{F} = F \) on \( C([0,1]) \) and \( \|\tilde{F}\|_{L^\infty([0,1])^*} = F_{C([0,1])} \). Prove that there exists no \( g \in L^1([0,1]) \) such that \( \tilde{F}(f) = \int_0^1 f(x)g(x) \, dx \) for all \( f \in L^\infty([0,1]) \).

3. Let \( \mathcal{X} \) be a normed vector space and \( \mathcal{X}^* \) its dual space.

   (1) Suppose \( x_n \to x \) weakly in \( \mathcal{X} \) (i.e., \( f(x_n) \to f(x) \) for any \( f \in \mathcal{X}^* \)). Prove that \( \sup_{n \geq 1} \|x_n\| < \infty \).

   (2) Assume in addition that \( \mathcal{X} \) is a Banach space. Suppose \( f_n \to f \) weak-* in \( \mathcal{X}^* \) (i.e., \( f_n(x) \to f(x) \) for any \( x \in \mathcal{X} \)). Prove that \( \sup_{n \geq 1} \|f_n\| < \infty \).

4. Let \( \mathcal{X} \) be an infinite-dimensional normed vector space. Prove that there exist \( x_k \in \mathcal{X} \) (\( k = 1, 2, \ldots \)) such that \( \|x_k\| = 1 \) for all \( k \geq 1 \) and \( \|x_j - x_k\| \geq 1/2 \) for all \( j, k = 1, 2, \ldots \) with \( j \neq k \). (See the hint in Exercise 19 on page 160.)

5. Let \( \mathcal{X} \) be a Banach space. Assume \( \mathcal{X}^* \) is separable. Prove that \( \mathcal{X} \) is also separable. (See the hint in Exercise 25 on page 160.)

6. Let \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) be two norms on a vector space \( \mathcal{X} \) such that \( \|x\|_1 \leq \|x\|_2 \) for any \( x \in \mathcal{X} \). Assume that \( \mathcal{X} \) is complete with respect to both norms. Prove that these norms are equivalent.

7. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces. Let \( T : \mathcal{X} \to \mathcal{Y} \) be a linear map such that \( f \circ T \in \mathcal{X}^* \) for every \( f \in \mathcal{Y}^* \). Prove that \( T \) is bounded.

8. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces. Let \( T_n \in L(\mathcal{X}, \mathcal{Y}) \) (\( n = 1, 2, \ldots \)) be such that \( \lim_{n \to \infty} T_n x \) exists for every \( x \in \mathcal{X} \). Define \( T x = \lim_{n \to \infty} T_n x \) (\( x \in \mathcal{X} \)). Prove that \( T \in L(\mathcal{X}, \mathcal{Y}) \).