1. Let $A$ and $B$ be two subsets of a topological space $X$. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Let $X$ be a topological space, $U$ an open subset of $X$, and $A$ a dense subset of $X$. Prove that $\overline{U} = U \cap \overline{A}$.

3. Prove that every separable metric space is second countable.

4. Prove that any metric space $(X, \rho)$ is normal, i.e., for any disjoint closed sets $A$ and $B$ of $X$, there are disjoint open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

5. Let $X$ and $Y$ be two topological spaces. Let $f : X \to Y$ be given. Prove that the following are equivalent:
   1. $f : X \to Y$ is continuous;
   2. $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$;
   3. $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$ for all $B \subseteq Y$.

6. Let $X$ be a topological space and $A \subseteq X$ a closed subset. Assume that $g \in C(A)$ satisfies $g = 0$ on $\partial A$. Prove that the extension of $g$ to $X$ defined by $g(x) = 0$ for $x \in A^c$ is continuous.

7. Let $X$ be a topological space and $Y$ a Hausdorff space. Let $f$ and $g$ be continuous maps from $X$ to $Y$. Prove the following:
   1. The set $\{x \in X : f(x) = g(x)\}$ is closed subset of $X$;
   2. If $f = g$ on a dense subset of $X$, then $f = g$ on all of $X$.

8. Prove the following:
   1. If $X_n (n = 1, 2, \ldots)$ are first countable topological spaces, then the product space $\prod_{n=1}^{\infty} X_n$ is also first countable;
   2. If $X_n (n = 1, 2, \ldots)$ are second countable topological spaces, then the product space $\prod_{n=1}^{\infty} X_n$ is also second countable.

9. Let $X$ be a topological space, $(Y, \rho)$ a complete metric space, and $\{f_n\}_{n=1}^{\infty}$ a sequence of maps from $X$ to $Y$. Assume that $\sup_{x \in X} \rho(f_n(x), f_m(x)) \to 0$ as $m, n \to \infty$. Prove that there is a unique map $f : X \to Y$ such that $\sup_{x \in X} \rho(f_n(x), f(x)) \to 0$ as $n \to \infty$. Moreover, if each $f_n$ is continuous, so is $f$. 