Math 240C: Real Analysis, Spring 2016

Homework Assignment 4

Due Monday, May 2, 2016

1. Compute the convolution $f * g$ with

$$f(t) = \begin{cases} e^t & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} t & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

2. Let $f(x) = (1/2) - x$ on $[0, 1]$ and extend f to be periodic on $\mathbb{R}$. Prove the following:
   (1) $\hat{f}(0) = 0$ and $\hat{f}(k) = (2\pi ik)^{-1}$ if $k \neq 0$;
   (2) $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.

   (See some hint for Problem 13 on page 254.)

3. **(Wirtinger's Inequality)** Let $a < b$. If $f \in C^1([a, b])$ and $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx.$$

   (See some hint for Problem 14 on page 254.)

4. Let $f \in L^1(\mathbb{R}^{n+m})$. Define

$$Pf(x) = \int_{\mathbb{R}^m} f(x, y)\, dy \qquad \forall x \in \mathbb{R}^n.$$

   Prove that $Pf \in L^1(\mathbb{R}^n)$, $\|Pf\|_1 \leq \|f\|_1$, and $(Pf)\hat{}(\xi) = \hat{f}(\xi, 0)$.

5. Define $\operatorname{sinc} x = (\sin \pi x)/\pi x$ if $x \neq 0$ and 1 if $x = 0$. Let $a > 0$. Prove that

$$\hat{\chi}_{[-a,a]}(x) = \chi^{\vee}_{[-a,a]}(x) = 2a \operatorname{sinc} 2ax.$$

6. Let $f_k = \chi_{[-1,1]} * \chi_{[-k,k]}$.
   (1) Compute $f_k(x)$ explicitly and show that $\|f\|_u = 2$.
   (2) Show that $f_k^{\vee}(x) = (\pi x)^{-2} sin 2\pi kx \sin 2\pi x$ and $\|f_k^{\vee}\|_1 \to \infty$ as $k \to \infty$.
   (3) Show that $\mathcal{F}(L^1)$ is a proper subset of C_0.

   (See some hints for problem 16 on page 255.)