1. Let $A = (a_{jk})$ be a real $n \times n$ matrix. Let $u \in C^2(\mathbb{R}^n)$ be such that $u \in L^1(\mathbb{R}^n)$, \( \partial_j u \in L^1(\mathbb{R}^n) \ (j = 1, \ldots, n), \) \( u(\infty) = 0, \) and \( \partial_x j u(\infty) = 0 \ (j = 1, \ldots, n). \) Define

\[
L_A u = - \sum_{j,k=1}^{n} a_{jk} \partial_2 x_j x_k u.
\]

Prove that \( (L_A u)(\xi) = 4 \pi^2 \sum_{j,k=1}^{n} a_{jk} \xi_j \xi_k \hat{u}(\xi) \quad \forall \xi \in \mathbb{R}^n. \)

2. Let $H$ be a Hilbert space and $M$ a dense subspace of $H$. Prove that any unitary isomorphism on $M$ can be uniquely extended to a unitary isomorphism on $H$.

3. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $f'$ exists point-wise on $\mathbb{R}$ and $f' \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Prove the following:

(1)

\[
\left[ \int |f(x)|^2 dx \right]^2 \leq 4 \int |xf(x)|^2 dx \int |f'(x)|^2 dx;
\]

(2) **(Heisenberg’s Inequality)** For any $b, \beta \in \mathbb{R},$

\[
\int (x-b)^2 |f(x)|^2 dx \int (\xi-\beta)^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{\|f\|_4^4}{16 \pi^2}.
\]

(See Problem 18 on page 255.)

4. Let $f \in L^1(\mathbb{R}^2)$ be radial, i.e., there exists $g : [0, \infty) \to \mathbb{R}$ such that $f(x) = g(|x|)$ for all $x \in \mathbb{R}^2$. Prove that $\hat{f}$ is also radial. (Note that this result is true for $\mathbb{R}^n$ for a general $n \geq 1$. See Problem 22 on page 256. Here, for $n = 2$, you can use the polar coordinates and change of variables.)

5. Let $0 < r < 1$. Consider the Poisson kernel on $\mathbb{T}$:

\[
P_r(x) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi ikx}.
\]

(1) Prove that

\[
P_r(x) = \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi x}.
\]

(2) Let $f \in L^1(\mathbb{T})$ and define

\[
A_r f(x) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{2\pi ikx}.
\]

Prove that $A_r f = f * P_r$. 