Math 240C: Real Analysis, Spring 2016

Homework Assignment 5

Due Monday, May 9, 2016

1. Let $A = (a_{jk})$ be a real $n \times n$ matrix. Let $u \in C^2(\mathbb{R}^n)$ be such that u and all its first-order and second-order partial derivatives are in $L^1(\mathbb{R}^n)$, and that u and all its first-order partial derivatives vanish at ∞. Define
$$L_A u = -\sum_{j,k=1}^{n} a_{jk} \partial^2_{x_j x_k} u.$$
Prove that
$$(L_A u)\hat{}(\xi) = 4\pi^2 \sum_{j,k=1}^{n} a_{jk} \xi_j \xi_k \hat{u}(\xi) \qquad \forall \xi \in \mathbb{R}^n.$$

2. Let H be a Hilbert space and M a dense subspace of H. Prove that any unitary isomorphism on M can be uniquely extended to a unitary isomorphism on H.

3. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that f' exists point-wise on $\mathbb{R}$ and $f' \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$. Prove the following:

   (1)
$$\left[\int |f(x)|^2 dx\right]^2 \leq 4 \int |xf(x)|^2 dx \int |f'(x)|^2 dx;$$

   (2) **(Heisenberg's Inequality)** For any b, $\beta \in \mathbb{R}$,
$$\int (x-b)^2 |f(x)|^2 dx \int (\xi - \beta)^2 |\hat{f}(\xi)|^2 d\xi \geq \frac{\|f\|_2^4}{16\pi^2}.$$

   (See Problem 18 on page 255.)

4. Let $f \in L^1(\mathbb{R}^2)$ be radial, i.e., there exists $g : [0, \infty) \to \mathbb{R}$ such that $f(x) = g(|x|)$ for all $x \in \mathbb{R}^2$. Prove that $\hat{f}$ is also radial. (Note that this result is true for $\mathbb{R}^n$ for a general $n \geq 1$. See Problem 22 on page 256. Here, for $n = 2$, you can use the polar coordinates and change of variables.)

5. Let $0 < r < 1$. Consider the Poisson kernel on $\mathbb{T}$:
$$P_r(x) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{2\pi i k x}.$$

   (1) Prove that
$$P_r(x) = \frac{1 - r^2}{1 + r^2 - 2r \cos 2\pi x}.$$

   (2) Let $f \in L^1(\mathbb{T})$ and define
$$A_r f(x) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{2\pi i k x}.$$
   Prove that $A_r f = f * P_r$.