1. Let \( U \) be a nonempty open subset of \( \mathbb{R}^n \). For any integer \( m \geq 0 \) and any \( \phi \in \mathcal{D}(U) \), let 
\[
\|\phi\|_m = \max_{|\alpha| \leq m} \sup_{x \in U} |\partial^\alpha \phi(x)|.
\]
This defines a norm on \( \mathcal{D}(U) \).

   (1) Prove that the family of norms \( \{\|\cdot\|_m\}_{m=0}^\infty \) defines \( \mathcal{D}(U) \) a locally convex Hausdorff space that is in fact a metric space.

   (2) For the case \( n = 1 \) and \( U = \mathbb{R} \), pick \( \phi \in \mathcal{D}(\mathbb{R}) \) with \( \phi > 0 \) on \( (0,1) \) and \( \phi = 0 \) outside \( (0,1) \). Define \( \phi_m(x) = \sum_{j=1}^m \phi(x-j)/j \) \( (m = 1, 2, \ldots) \). Prove that, with respect to the topology defined in part (1), \( \{\phi_m\}_{m=1}^\infty \) is a Cauchy sequence but it does not converge to any function in \( \mathcal{D}(\mathbb{R}) \).

2. Let \( U \) be a nonempty open subset of \( \mathbb{R}^n \). For any \( f \in L^1_{\text{loc}}(U) \), define 
\[
T_f(\phi) = \int_{\mathbb{R}^n} f(x) \phi(x) \, dx \quad \forall \phi \in \mathcal{D}.
\]

   (1) Prove \( T_f \) is a distribution on \( U \).

   (2) Let \( 1 \leq p \leq \infty \). Assume all \( f_j \) \( (j = 1, 2, \ldots) \) and \( f \) are in \( L^p(U) \) and \( f_j \to f \) in \( L^p(U) \). Prove that \( T_{f_j} \to T_f \) in \( \mathcal{D}'(U) \).

   (3) Consider \( n = 1 \) and \( U = \mathbb{R} \). Construct \( L^1_{\text{loc}}(\mathbb{R}) \)-functions \( f_j \) \( (j = 1, 2, \ldots) \) and \( f \) such that \( f_j \to f \) pointwise on \( \mathbb{R} \) but \( \{T_{f_j}\} \) does not converge to \( T_f \) in \( \mathcal{D}'(\mathbb{R}) \).

3. Let \( U \) be a nonempty open subset of \( \mathbb{R}^n \) and \( \mu \) be a Radon measure on \( U \). Define 
\[
T_\mu(\phi) = \int_U \phi(x) \, d\mu \quad \forall \phi \in \mathcal{D}(U).
\]

   Prove that \( T_\mu \) is a distribution on \( U \) and that \( \text{supp}(T_\mu) = \text{supp}(\mu) \).

4. Suppose that \( f \) is continuously differentiable on \( \mathbb{R} \) except at \( x_1, \ldots, x_m \), where \( f \) has jump discontinuities, and that its pointwise derivative \( df/dx \) (defined except at \( x_1, \ldots, x_m \)) is in \( L^1_{\text{loc}}(\mathbb{R}) \). Prove that the distributional derivative of \( f \) is given by 
\[
f' = (df/dx) + \sum_{j=1}^m [f(x_j+) - f(x_j-)] \tau_{x_j} \delta.
\]

5. Let \( H \) be the Heaviside function on \( \mathbb{R} \): \( H(x) = 1 \) if \( x > 0 \) and \( H(x) = 0 \) if \( x \leq 0 \). Let \( \delta \) be the Dirac measure at 0 (identified as a distribution on \( \mathbb{R} \)). Finally, let 1 be the constant function whose value is 1 at any point in \( \mathbb{R} \), identified as a locally integrable function and further as a distribution (cf. Problem 2). Prove that \( \delta' \ast H = \delta \), \( 1 \ast \delta' = 0 \), and the associative law fails: \( 1 \ast (\delta' \ast H) \neq (1 \ast \delta') \ast H \).

6. (Optional) Problem 11 on page 290.

7. Problem 20 on page 299.