Math 240C: Real Analysis, Spring 2016

Homework Assignment 6

Due Monday, May 23, 2016

1. Let U be a nonempty open subset of $\mathbb{R}^n$. For any integer $m \geq 0$ and any $\phi \in \mathcal{D}(U)$, let $\|\phi\|_m = \max_{|\alpha| \leq m} \sup_{x \in U} |\partial^\alpha \phi(x)|$. This defines a norm on $\mathcal{D}(U)$.
   (1) Prove that the family of norms $\{\|\cdot\|_m\}_{m=0}^\infty$ defines $\mathcal{D}(U)$ a locally convex Hausdorff space that is in fact a metric space.
   (2) For the case $n = 1$ and $U = \mathbb{R}$, pick $\phi \in \mathcal{D}(\mathbb{R})$ with $\phi > 0$ on $(0, 1)$ and $\phi = 0$ outside $(0, 1)$. Define $\phi_m(x) = \sum_{j=1}^m \phi(x - j)/j$ $(m = 1, 2, \dots)$. Prove that, with respect to the topology defined in part (1), $\{\phi_m\}_{m=1}^\infty$ is a Cauchy sequence but it does not converge to any function in $\mathcal{D}(\mathbb{R})$.

2. Let U be a nonempty open subset of $\mathbb{R}^n$. For any $f \in L^1_{\text{loc}}(U)$, define

$$T_f(\phi) = \int_{\mathbb{R}^n} f\phi \, dx \qquad \forall \phi \in \mathcal{D}.$$

   (1) Prove T_f is a distribution on U.
   (2) Let $1 \leq p \leq \infty$. Assume all f_j $(j = 1, 2 \dots)$ and f are in $L^p(U)$ and $f_j \to f$ in $L^p(U)$. Prove that $T_{f_j} \to T_f$ in $\mathcal{D}'(U)$.
   (3) Consider $n = 1$ and $U = \mathbb{R}$. Construct $L^1_{\text{loc}}(\mathbb{R})$-functions f_j $(j = 1, 2 \dots)$ and f such that $f_j \to f$ pointwise on $\mathbb{R}$ but $\{T_{f_j}\}$ does not converge to T_f in $\mathcal{D}'(\mathbb{R})$.

3. Let U be a nonempty open subset of $\mathbb{R}^n$ and μ be a Radon measure on U. Define

$$T_\mu(\phi) = \int_U \phi \, d\mu \qquad \forall \phi \in \mathcal{D}(U).$$

   Prove that T_μ is a distribution on U and that $\operatorname{supp}(T_\mu) = \operatorname{supp}(\mu)$.

4. Suppose that f is continuously differentiable on $\mathbb{R}$ except at $x_1, \dots, x_m$, where f has jump discontinuities, and that its pointwise derivative df/dx (defined except at $x_1, \dots, x_m$) is in $L^1_{\text{loc}}(\mathbb{R})$. Prove that the distributional derivative of f is given by $f' = (df/dx) + \sum_{j=1}^m [f(x_j+) - f(x_j-)]\tau_{x_j}\delta$.

5. Let H be the Heaviside function on $\mathbb{R}$: $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x \leq 0$. Let δ be the Dirac measure at 0 (identified as a distribution on $\mathbb{R}$). Finally, let 1 be the constant function whose value is 1 at any point in $\mathbb{R}$, identified as a locally integrable function and further as a distribution (cf. Problem 2). Prove that $\delta' * H = \delta$, $1 * \delta' = 0$, and the associative law fails: $1 * (\delta' * H) \neq (1 * \delta') * H$.

6. (Optional) Problem 11 on page 290.

7. Problem 20 on page 299.