Monday, 4/20/2020, Lecture 10

Review. $X, Y$: LCH, $\mu, \nu$: Radon meas. on $(X, \mathcal{B}_X)$, $(Y, \mathcal{B}_Y)$.

- $X, Y$: 2nd countable $\Longrightarrow$ $\mathcal{B}_{X\times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$, $\mu\times\nu$ is Radon on $(X\times Y, \mathcal{B}_{X\times Y})$.
- In general, $I(f) = \int f\, d(\mu\times\nu)$, $\forall f \in C_c(X\times Y)$ defines the product Radon meas. $\mu \hat{\times} \nu$ on $(X\times Y, \mathcal{B}_{X\times Y})$:

$$\int f\, d(\mu\hat{\times}\nu) = \int f\, d(\mu\times\nu) \quad \forall f \in C_c(X\times Y).$$

Thm 7.26 $X, Y$: LCH, $\mu, \nu$: $\sigma$-finite Radon meas. on $X, Y$, resp. $E \in \mathcal{B}_{X\times Y}$. Then

- $E_x \in \mathcal{B}_Y$ $(\forall x\in X)$, $E^y \in \mathcal{B}_X$ $(\forall y \in Y)$,
- $x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$: Borel-meas. on $X$, $Y$ resp.,
- $\mu\hat{\times}\nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$ ........ $\star$

Moreover, $\mu\hat{\times}\nu\big|_{\mathcal{B}_X\otimes\mathcal{B}_Y} = \mu\times\nu$.

Pf If $\star$ is true, then $E \in \mathcal{B}_X\otimes\mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y} \overset{\text{Thm 2.36}}{\Longrightarrow} \mu\times\nu(E) = \int \nu(E_x)\, d\mu(x) \overset{\star}{=} \mu\hat{\times}\nu(E)$. Hence $\mu\hat{\times}\nu = \mu\times\nu$ on $\mathcal{B}_X\otimes\mathcal{B}_Y$.

Proceed now in two steps.

Step1 Assume $U, V$ open in $X, Y$, $\mu(U)<\infty$, $\nu(V)<\infty$.
Let $W=U\times V$. Def. $\mathcal{M} = \{E\in\mathcal{B}_{X\times Y} : E\cap W$ satisfies the conclusions of the thm$\} \subseteq \mathcal{B}_{X\times Y}$. We show $\mathcal{M}=\mathcal{B}_{X\times Y}$.

We have

① Prop. 7.25 $\Longrightarrow$ $\mathcal{T}_{X\times Y} \subseteq \mathcal{M}$.

② $E,F\in\mathcal{M}$, $F\subseteq E \Longrightarrow E\setminus F\in\mathcal{M}$. $E\in\mathcal{M}\Rightarrow E^c = X\setminus E\in\mathcal{M}$.

Indeed: $\mu\hat{\times}\nu(E\cap W) = \mu\hat{\times}\nu(F\cap W) + \mu\hat{\times}\nu((E\setminus F)\cap W)$.
$\nu((E\cap W)_x) = \nu((F\cap W)_x) + \nu(((E\setminus F)\cap W)_x)$, $\mu((E\cap W)^y) = \cdots$
All measures are finite. So, subtractions ok. Conclusions true for $E, F$, subtraction $\Longrightarrow$ also true for $E\setminus F$. i.e., $E\setminus F\in\mathcal{M}$.

③ $\mathcal{M}$ is closed under finite disjoint unions (by the additivity of measures).

④ $\mathcal{M}$ is closed under countable increasing unions, and hence, by ②, under countable decreasing intersections (by the monotone conv. thm).

Now, let $\Sigma = \{A \setminus B : A, B \text{ open in } X \times Y\}$. Let $\mathcal{A}$ be the collection of finite disj. unions of sets in $\Sigma$.
Then $\mathcal{E}$ is an elementary family:

- $\phi \in \Sigma$;
- $E = A_1 \setminus B_1,\ F = A_2 \setminus B_2 \in \Sigma \Rightarrow E \cap F = (A_1 \cap A_2) \setminus (B_1 \cup B_2) \in \Sigma$;
- $E = A \setminus B \in \Sigma \Longrightarrow E^c = (X \times Y \setminus A) \cup (A \cap B \setminus \phi) \in \mathcal{A}$.

Thus, $\mathcal{A}$ is an alg. (by Prop. 1.7). Lemma 2.35 (The Monoton Class Lemma) $\Longrightarrow$ the monotone class generated by $\mathcal{A}$ = the $\sigma$-alg. genenerated by $\mathcal{A}$. But ① – ④ (Note: if $A \setminus B \in \Sigma$ then $A \setminus B = A \setminus (A \cap B) \in \mathcal{M}$) $\Longrightarrow \mathcal{M}$ contains this monotone class, which $\supseteq \Sigma \supseteq \mathcal{T}_{X \times Y}$. So, $\mathcal{M} = \mathcal{B}_{X \times Y}$.

Step 2 If $\mu, \nu$ are $\sigma$-finite, then $X = \bigcup_1^\infty U_n$, $Y = \bigcup_1^\infty V_n$: $U_n, V_n$: open in $X, Y$, $U_n \uparrow$, $V_n \uparrow$, $\mu(U_n) < \infty$, $\nu(V_n) < \infty$. (If $\mu(E) < \infty$ then, the outer reg. $\Rightarrow \exists$ open $U \supseteq E$: $\mu(U) < \infty$.) $\forall E \in \mathcal{B}_{X \times Y} \Longrightarrow$ All $E \cap (U_n \times V_n)$ satisfy the conclusions. $\Longrightarrow$ E also satisfies the conclusions by ~~the~~ MCT. QED

Thm 7.27. (The Fubini-Tonelli Thm for Radon Products)
$X, Y$: LCH, $\mu, \nu$: $\sigma$-finite Radon measures on $X, Y$, resp. $f: X \times Y \to \mathbb{C}$. Borel measurable. Then

- $f_x, f^y$ are Borel-measurable $\forall x, y$.
- If $f \geq 0$ then $x \mapsto \int f_x \, d\nu$, $y \mapsto \int f^y \, d\mu$ are Borel-measurable.

If $f \in L^1(\mu \hat{\times} \nu)$ then $f_x \in L^1(\nu)$ a.e. $x$, $f^y \in L^1(\mu)$, a.e. $y$, and $x \mapsto \int f_x \, d\nu$, $y \mapsto \int f^y \, d\mu$ are in $L^1(\mu)$, $L^1(\nu)$.

In both cases: $\int f \, d\mu \hat{\times} \nu = \iint f \, d\mu \, d\nu = \iint f \, d\nu \, d\mu$.

Pf The measurability of $f_x, f^y$ follow Lemma 7.23.
The rest of the pf is the same as before, except using Thm 7.26 instead of Thm 2.36. QED

Generalization to a family of Radon measures.

$(X_\alpha, \mathcal{B}_{X_\alpha}, \mu_\alpha)$, $\alpha \in \mathcal{A}$. We assume

- $X_\alpha$ : Compact Hausdorff, $\forall \alpha \in \mathcal{A}$.
- $\mathcal{B}_{X_\alpha}$ : Borel $\sigma$-alg. on $X_\alpha$.
- $\mu_\alpha$ : Radon measure on $X_\alpha$ s.t. $\mu_\alpha(X_\alpha) = 1$.

Def. $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$. Tychonoff's Thm $\Longrightarrow$ $X$ is compact (and Hausdorff).

$\forall \alpha_1, \dots, \alpha_n \in \mathcal{A}$. Def. $\pi_{(\alpha_1, \dots, \alpha_n)} : X \to \prod_{j=1}^{n} X_{\alpha_j}$ : $\pi_{(\alpha_1,\dots,\alpha_n)}(x) = (x_{\alpha_1}, \dots, x_{\alpha_n})$

Thus, $E_{\alpha_j} \in \mathcal{B}_{X_{\alpha_j}}$ $(1 \le j \le n)$ $\Longrightarrow$ $\pi^{-1}_{(\alpha_1,\dots,\alpha_n)}(E_{\alpha_1} \times \dots \times E_{\alpha_n}) = \prod_{\alpha \in \mathcal{A}} E_\alpha$ where $E_\alpha = X_\alpha$ if $\alpha \ne \alpha_j$ $(1 \le j \le n)$.

<u>Thm</u> Under these assumptions, there exists a unique Radon meas. on $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ s.t. $\forall \alpha_1, \dots, \alpha_n \in \mathcal{A}$, $\forall E \in \mathcal{B}_{\prod_{j=1}^{n} X_{\alpha_j}}$,

$$\mu\left(\pi^{-1}_{(\alpha_1,\dots,\alpha_n)}(E)\right) = (\mu_{\alpha_1} \hat{\times} \cdots \hat{\times} \mu_{\alpha_n})(E).$$