Wednesday, 4/22/2020, Lecture 11

Ch. 8 Elements of Fourier Analysis

Fourier analysis, or harmonic analysis is a fundamental subject / area of analysis with many applications (PDEs, approximation theory, numerical analysis, algebraic number theory, etc.).

Topics
- convolution
- Fourier transform
- Fourier series
- Fourier analysis of measures

Mainly: $\mathbb{R}^n$, Lebesgue measure

§8.1 Preliminaries

I. Notation
II. The Schwartz space
III. The translation operator

I. Notation

$U \subseteq \mathbb{R}^n$: open, $k \in \mathbb{N}$: $C^\infty(U) = \bigcap_{k=1}^{\infty} C^k(U)$

$C^k(U) = \{$all $f: U \to \mathbb{C}$ whose partial derivatives of order $\le k$ exist and are continuous$\}$

$E \subseteq \mathbb{R}^n$: $C_c^\infty(E) = \{f \in C^\infty(\mathbb{R}^n)$: supp$(f)$ is compact, and supp$(f) \subseteq E\}$

$$L^p(E) = L^p(m, E)$$

$L^p = L^p(\mathbb{R}^n)$, $C^k = C^k(\mathbb{R}^n)$, $C^\infty = C^\infty(\mathbb{R}^n)$, $C_c^\infty = C_c^\infty(\mathbb{R}^n)$.

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x| = \sqrt{x \cdot x}$, $x \cdot y = \sum_1^n x_j y_j$.

Multi-index: $\alpha = (\alpha_1, \dots, \alpha_n)$, each $\alpha_j \in \mathbb{Z}$, $\alpha_j \ge 0$.

$|\alpha| = \sum_1^n \alpha_j$, $\alpha! = \prod_1^n \alpha_j!$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

$\frac{\partial}{\partial x_j} = \partial_{x_j} = \partial_j$. e.g., $n = 4$, $\partial^{(1,0,2,7)} = \partial_{x_1} \partial_{x_3}^2 \partial_{x_4}^7$.

Taylor's expansion for $f \in C^k$:

$$f(x) = \sum_{|\alpha| \le k} (\partial^\alpha f)(x_0) \frac{(x-x_0)^\alpha}{\alpha!} + R_k(x), \quad \lim_{x \to x_0} \frac{|R_k(x)|}{|x-x_0|^k} = 0.$$

The product rule:

$$\partial^\alpha (f g) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\,\gamma!} \partial^\beta f \, \partial^\gamma g .$$

Examples ① $\eta(t) = e^{-1/t} \chi_{(0,\infty)}$. $\eta \in C^\infty(\mathbb{R})$

② $\psi(x) = \eta(1-|x|^2) = \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1, \\ 0 & \text{if } |x|>1. \end{cases}$

- $\psi \in C_c^\infty(\mathbb{R}^n)$
- $\psi \ge 0$ on $\mathbb{R}^n$
- $\operatorname{supp}(\psi) = \{x \in \mathbb{R}^n : |x| \le 1\} = \overline{B(0,1)}$
- ψ is radially symmetric

(II) The Schwartz space

$$\mathcal{S} = \{ f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \quad \forall N \quad \forall \alpha \}$$

$N \geq 0$: integer; $\alpha = (\alpha_1, \cdots, \alpha_n)$: multi-index.

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|.$$

⊙ $\|f\|_{(N,\alpha)} < \infty \iff |\partial^\alpha f(x)| \leq C_{N,\alpha} (1+|x|)^{-N} \quad \forall x \in \mathbb{R}^n$
for some const. $C_{N,\alpha} > 0$ (e.g., $C_{N,\alpha} = \|f\|_{(N,\alpha)}$).

$f \in \mathcal{S}$: $\forall \alpha$. $|\partial^\alpha f|$ decays at ∞ faster than $|x|^{-N}$ for any $N \geq 0$.

⊙ Example. $f(x) = |x|^\alpha e^{-|x|^2}$.

⊙ $C_c(\mathbb{R}^n) \subseteq \mathcal{S} \subseteq L^p(\mathbb{R}^n) \quad \forall 1 \leq p \leq \infty$.

$f \in \mathcal{S}$, $\forall \alpha \implies f^\alpha \in L^p \quad (1 \leq p \leq \infty)$.

$|\partial^\alpha f(x)|^p \leq C(1+|x|)^{-Np}$, choose N: $Np > n$.

<u>Prop 8.3</u> Let $f \in C^\infty$. The following are equivalent:

(1) $f \in S$;

(2) $\forall \alpha, \beta$: $x^\beta \partial^\alpha f$ is bounded;

(3) $\forall \alpha, \beta$: $\partial^\alpha(x^\beta f)$ is bounded.

<u>Pf</u> (1) $\Longrightarrow$ (2). $|x^\beta| \le (1+|x|)^N$ if $|\beta| \le N$.

(2) $\Longrightarrow$ (1) $\sum_1^n |x_j|^N > 0$ on $|x|=1$. $\Longrightarrow \exists a > 0$ s.t. $\sum_1^n |x_j|^N \ge a$ on $|x|=1$. $\Longrightarrow \forall x \in \mathbb{R}^n$: $\sum_1^N |x_j|^N \ge a|x|^N$.

Hence,

$$(1+|x|)^N \le 2^N(1+|x|^N) \le 2^N\left(1+a^{-1}\sum_1^n |x_j|^N\right) \le 2^N a^{-1} \sum_{|\beta| \le N} |x^\beta|.$$

(2) $\Longleftrightarrow$ (3) $x^\beta \partial^\alpha f$ is a linear combination of functions $\partial^\gamma(x^\delta f)$, vice versa. <u>QED</u>

Topology of $\mathcal{S}$

$\forall N, \alpha$: $f \longmapsto \|f\|_{(N,\alpha)}$ is a seminorm on $\mathcal{S}$. The <u>countable</u> family of seminorms $\{\|\cdot\|_{(N,\alpha)}\}_{\text{all } N, \alpha}$ meak $\mathcal{S}$ a topological vector space (TVS).

- The topology is generated by all
$$U_{(N,\alpha), f, \varepsilon} = \{g \in \mathcal{S} : \|g - f\|_{(N,\alpha)} < \varepsilon\}.$$
Any open set is ϕ or $\mathcal{S}$ or a union of finite intersections of such sets.
- $f_n \longrightarrow f$ in $\mathcal{S} \iff \|f_n - f\|_{(N,\alpha)} \to 0 \quad \forall N, \alpha$.
- $\mathcal{S}$ is Hausdorff, since all $\|f\|_{(N,\alpha)} = 0 \iff f = 0$.
  Therefore, this topology is metrizable.
- The metric space $\mathcal{S}$ is complete (see next page).

Prop. 8.2 $\mathcal{S}$ is a Fréchet space.

Fréchet space: complete Hausdorff TVS defined by a countable family of seminorms.

Pf Only completeness. Let $\{f_k\}$ be a Cauchy seq.
$\|f_k - f_j\|_{(N,\alpha)} \to 0$ $(k,j \to \infty)$ $\forall N, \alpha$. So, $\forall \alpha$. $\partial^\alpha f_k \to g_\alpha$ unif. for some $g_\alpha \in C(\mathbb{R}^n)$. Denote $e_j = (0, \cdots, 0, \underset{\underset{j}{\uparrow}}{1}, 0, \cdots, 0)$.

$$f_k(x+te_j) - f_k(x) = \int_0^t \partial_j f_k(x+se_j)\,ds. \quad \text{Let } k \to \infty.$$

$$g_0(x+te_j) - g_0(x) = \int_0^t g_{e_j}(x+se_j)\,ds.$$

By the fundamental thm of the calculus, $g_{e_j} = \partial_j g_0$.
Now, induction on $|\alpha|$ implies $g_\alpha = \partial^\alpha g_0$ $\forall \alpha$. Moreover, $\|f_k - g_0\|_{(N,\alpha)} \to 0$ as $k \to \infty$ $\forall \alpha$. QED

(III) <u>Translation</u> Define for $f : \mathbb{R}^n \to \mathbb{C}$ and $y \in \mathbb{R}^n$, the <u>translation</u>: $\tau_y f(x) = f(x-y) \quad \forall x \in \mathbb{R}^n$.

- $\forall 1 \le p \le \infty \quad \|\tau_y f\|_p = \|f\|_p \quad \forall y$.
- $\|\tau_y f\|_u = \|f\|_u, \ \forall y$.
- $\|\tau_y f - f\|_u \to 0 \iff \tau_y f \to f$ uniformly on $\mathbb{R}^n$.

<u>Lemma</u> $f \in C_c(\mathbb{R}^n) \Rightarrow f$ is uniformly continuous.

<u>Pf</u> Let $K = \mathrm{supp}(f)$: compact. $\forall \varepsilon > 0$, $\forall x \in K$.
f is cont. at $x \Longrightarrow \exists \delta_x > 0$ s.t. $|f(x-y) - f(x)| < \frac{1}{2}\varepsilon$ if $|y| < \delta_x$. $\{B(x, \frac{1}{2}\delta_x)\}_{x \in K}$ covers $K \Longrightarrow \exists x_1, \cdots, x_m$ s.t. $\{B(x_j, \frac{1}{2}\delta_{x_j})\}_{j=1}^m$ covers K. Let $\delta = \min(\frac{1}{2}\delta_{x_1}, \cdots, \frac{1}{2}\delta_{x_m}) > 0$.
Then, $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$ with $|y| < \delta$. If $x \in K$ then $x \in B(x_j, \frac{1}{2}\delta_j)$
so, $x - y \in B(x_j, \delta_{x_j})$. Hence $|f(x-y) - f(x)| \le |f(x-y) - f(x_j)|$
$+ |f(x_j + (x - x_j)) - f(x)| < \varepsilon$. If $x - y \in K$, then $x = x - y + y$,
the same argument applies. If $x-y, x \notin K$, then $f(x-y) = f(x) = 0$.

QED

Prop. 8.5 If $1 \le p < \infty$, $f \in L^p$ then $\|\tau_y f - f\|_p \to 0$ as $y \to 0$.
In particular, $\forall z \in \mathbb{R}^n$, $\|\tau_{y+z} f - \tau_z f\| \to 0$ as $y \to 0$.
(as $\tau_{y+z} f = \tau_y \tau_z f$). i.e., the translation is cont. in the L^p norm.

Pf Let $g \in C_c$. $\forall |y| \le 1$, $\tau_y g \in C_c$ are all supported on a common compact set K. Thus.

$$\int |\tau_y g - g|^p \le \|\tau_y g - g\|_u^p \, m(K) \to 0 \quad \text{as } y \to 0.$$

Now, let $f \in L^p$, $\forall \varepsilon > 0$. $\exists g \in C_c$ s.t. $\|g - f\|_p < \varepsilon$ (Prop. 7.9),

So,
$$\begin{aligned} \|\tau_y f - f\|_p &\le \|\tau_y (f-g)\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &= \|f - g\|_p + \|\tau_y g - g\|_p + \|g - f\|_p \\ &< 2\varepsilon + \|\tau_y g - g\|_p \end{aligned}$$

Thus, $\limsup_{y \to 0} \|\tau_y f - f\|_p \le 2\varepsilon$. i.e., $\|\tau_y f - f\|_p \to 0$ as $y \to 0$.

QED