Friday, 4/24/2020. Lecture 12

§8.2 Convolutions

Def $f, g: \mathbb{R}^n \to \mathbb{C}$, measurable. the convolution is

$$(f * g)(x) = \int f(x-y)\, g(y)\, dy \quad x \in \mathbb{R}^n, \text{ if integrable.}$$

- $f * g$ is a.e. defined if, e.g., f is bounded and compactly supported, and $g \in L^1_{loc}(\mathbb{R}^n)$.
- Given $f: \mathbb{R}^n \to \mathbb{C}$, def. $K: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by $K(x-y) = f(x-y)$.

  f is Borel meas. on $\mathbb{R}^n \Longrightarrow K$ is Borel meas. on $\mathbb{R}^n \times \mathbb{R}^n$.

  f is Lebesgue meas. on $\mathbb{R}^n \Longrightarrow K$ is Lebesgue meas. on $\mathbb{R}^n \times \mathbb{R}^n$.

Topics
- Basic operational properties.
- L^p-property, e.g., $\|f * g\|_p \le \|f\|_1 \|g\|_p$
- Differentiability. $\partial^\alpha (f * g) = (\partial^\alpha f) * g$. $f, g \in \mathcal{S} \Longrightarrow f * g \in \mathcal{S}$
- Approximations (molifiers)

Prop. 8.6 Assume all integrals exist.

(a) $f*g = g*f$

$$(f*g)(x) = \int f(x-y)\,g(y)\,dy \overset{z=x-y}{=\!=\!=} \int f(z)\,g(x-z)\,dz = (g*f)(x)$$

(b) $(f*g)*h = f*(g*h)$

$$(f*g)*h(x) = \int (f*g)(x-y)\,h(y)\,dy \overset{(a)}{=} \iint f(z)\,g(x-y-z)\,h(y)\,dz\,dy$$

$$\overset{\text{Fubini}}{=} \int f(z)\,(g*h)(x-z)\,dz \overset{(a)}{=} f*(g*h)(x)$$

(c) $\forall z \in \mathbb{R}^n$, $\tau_z(f*g) = (\tau_z f)*g = f*\tau_z g$.

$$\tau_z(f*g)(x) = (f*g)(x-z) = \int f(x-z-y)\,g(y)\,dy$$

$$= \int \tau_z f(x-y)\,g(y)\,dy = (\tau_z f)*g(x).$$

(d) $\operatorname{supp}(f*g) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$

$$= \overline{\{x+y : x \in \operatorname{supp}(f),\ y \in \operatorname{supp}(g)\}} =: A.$$

$z \notin A \Rightarrow \forall y \in \operatorname{supp}(g)$, $z-y \notin \operatorname{supp}(f) \Rightarrow f(z-y)\,g(y) = 0$ $\forall y$. So, $(f*g)(z) = \int f(z-y)\,g(y)\,dy = 0$. QED

<u>Young's ineq.</u> $f \in L^1$, $g \in L^p$ $(1 \le p < \infty) \Longrightarrow f*g(x)$ exists a.e. x, $f*g \in L^p$, and $\|f*g\|_p \le \|f\|_1 \|g\|_p$.

<u>Pf</u> By Thm 6.18 with $K(x,y) = f(x-y)$. <u>QED</u>

<u>Prop. 8.8</u> Let $p, q \in [1, \infty]$ be conjugate, $f \in L^p$, and $g \in L^q$. Then $f*g(x)$ exists at any $x \in \mathbb{R}^n$, $f*g$ is bounded and unif. cont., and $\|f*g\|_u \le \|f\|_p \|g\|_q$.

If in addition $p, q \in (1, \infty)$, then $f*g \in C_0$

<u>Pf</u> Existence and $\|f*g\|_u \le \|f\|_p \|g\|_q$ by Hölder. If $1 \le p < \infty$,

$$\|\tau_y(f*g) - f*g\|_u = \|(\tau_y f - f) * g\|_\infty \le \|\tau_y f - f\|_p \|g\|_q \to 0 \text{ if } y \to 0.$$

(If $p = \infty$, exchang f, g.) If $1 < p, q < \infty$, choose $f_n, g_n \in C_c$ s.t. $\|f_n - f\|_p \to 0$, $\|g_n - g\|_q \to 0$. Then $f_n * g_n \in C_c$ and

$$\|f_n * g_n - f*g\|_u \le \|f_n - f\|_p \|g_n\|_q + \|f\|_p \|g_n - g\|_q \to 0.$$

So, $f*g \in C_0$. <u>QED</u>

<u>Prop. 8.9</u> Suppose $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

(a) (Young's ineq., general form) $f \in L^p$, $g \in L^q \Rightarrow f * g \in L^r$ and $\|f * g\|_r \le \|f\|_p \|g\|_q$.

(b) $p > 1$, $q > 1$, $r < \infty$, $f \in L^p$, $g \in \text{Weak } L^q \Rightarrow f * g \in L^r$ and $\|f * g\|_r \le C_{pq} \|f\|_p [g]_q$ (the const. C_{pq} is indep. of f, g).

(c) $p = 1$, $r = q > 1$, $f \in L^1$, $g \in \text{weak } L^q \Rightarrow f * g \in \text{weak } L^q$ and $[f * g]_q \le C_q \|f\|_1$ (the const. C_q is indep. of f, g).

<u>Pf</u> (a) $p = 1$, $r = q$: Young's ineq. $p = q/(q-1)$, $r = \infty$: Prop. 8.8.

Fix q. Use the generalized Hölder's ineq. to get

$$|f * g(x)|^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q \, dy.$$

Hence $\|f * g\|_r^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \|f\|_p^p \|g\|_q^q = \|f\|_p^r \|g\|_q^r$.

(b) and (c): special case of Thm 6.36. <u>QED</u>