Monday, 4/27/2020, Lecture 13.

§8.2 Convolutions (Cont'd)

Differentiability / Smoothness of $f*g$.

$$\partial_x^\alpha (f*g)(x) = \partial_x^\alpha \int f(x-y)\,g(y)\,dy = \int \partial_x^\alpha f(x-y)\,dy = (\partial^\alpha f)*g.$$

Prop. 8.10 $f \in C^1$, $g \in C^k$, and all $\partial^\alpha g$ ($|\alpha| \le k$) are bded.

$\Longrightarrow$ $f*g \in C^k$ and $\partial^\alpha (f*g) = (\partial^\alpha f)*g$ $\forall |\alpha| \le k$.

Pf. By Thm 2.27: exchange differentiation & integration.

Prop. 8.11 $f, g \in \mathcal{S} \Longrightarrow f*g \in \mathcal{S}$.

Pf. $f*g \in C^\infty$ by Prop. 8.10. Note that

$$1+|x| \le 1 + |x-y| + |y| \le (1+|x-y|)(1+|y|) \quad \forall x, y \in \mathbb{R}^n.$$

So, for any $N \ge 0$ (integer), any α (multi-index),

$$(1+|x|)^N |\partial^\alpha (f*g)(x)| \le \int (1+|x-y|)^N |\partial^\alpha f(x-y)| \, (1+|y|)^N |g(y)|\,dy$$

$$\le \|f\|_{(N,\alpha)} \int (1+|y|)^{N+n+1} |g(y)| \, (1+|y|)^{-n-1}\,dy$$

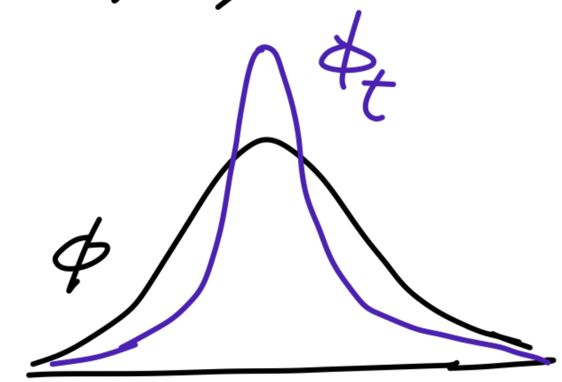
$$\le \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \int (1+|y|)^{-n-1}\,dy < \infty.$$

QED

Approximations

Let $\phi \in L^1(\mathbb{R}^n)$. Def. $\phi_t(x) = \frac{1}{t^n}\phi\left(\frac{x}{t}\right)$, $x \in \mathbb{R}^n$, $t > 0$.

Then $\int \phi_t \, dx = \int \phi \, dx$ $\forall t > 0$.



Thm 8.14 Let $\phi \in L^1$ and $\int \phi \, dx = 1$.

(1) $f \in L^p$ $(1 \le p < \infty) \Longrightarrow f * \phi_t \longrightarrow f$ in L^p as $t \to 0$.

(2) $f \in L^\infty$ and f is uniformly cont. $\Longrightarrow f * \phi_t \to f$ uniformly as $t \to 0$.

(3) $f \in L^\infty$ and f is cont. on an open set $U \Longrightarrow$ $f * \phi_t \longrightarrow f$ uniformly on compact subsets of U as $t \to 0$.

Call $\{\phi_t\}_{t>0}$ an approximate identity. $\boxed{\int \phi_t = 1}$

Pf of Thm (1) $(f * \phi_t)(x) - f(x) = \int [f(x-y) - f(x)] \, \phi_t(y) \, dy$

$\overset{y=tz}{=} \int [f(x - tz) - f(x)] \, \phi(z) \, dz = \int [\tau_{tz} f(x) - f(x)] \phi(z) \, dz.$

$$|(f*\phi_t)(x)-f(x)| \le \int |\tau_{tz}f(x)-f(x)|\,|\phi(z)|^{1/p}\,|\phi(z)|^{1/q}\,dz$$

$$\le \left(\int |\tau_{tz}f(x)-f(x)|^p|\phi(z)|\,dz\right)^{1/p}\underbrace{\left(\int|\phi(z)|\,dz\right)^{1/q}}_{=1} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right)$$

$$\|f*\phi_t-f\|_p^p \le \iint |\tau_{tz}f(x)-f(x)|^p|\phi(z)|\,dz\,dx = \int \|\tau_{tz}f-f\|_p^p\,|\phi(z)|\,dz$$

$\xrightarrow{DCT} 0$ as $\|\tau_{tz}f-f\|_p \le 2\|f\|_p$ and $\|\tau_{tz}f-f\|_p \to 0$ as $t\to 0$ for each z (Prop. 8.5).

2 $|(f*\phi_t)(x)-f(x)| \le \int |\tau_{tz}f(x)-f(x)|\,|\phi(z)|\,dz.$

$\|f*\phi_t-f\|_u \le \int \|\tau_{tz}f-f\|_u\,|\phi(z)|\,dz \xrightarrow{DCT} 0$ as $t\to 0$.

3 $\forall \varepsilon>0$. Since $\phi\in L^1$, there exists a compact set $F\subseteq \mathbb{R}^n$ such that $\int_{F^c}|\phi|\,dx<\varepsilon$. Let K be a compact subset of U. If $0<t<<1$ then $x-tz\in U$ $\forall x\in K$ $\forall z\in F$. Thus, $\sup_{x\in K,\, z\in F}|f(x-tz)-f(x)|<\varepsilon$ $(0<t<<1)$.

Thus, $\sup_{x\in K}|(f*\phi_t)(x)-f(x)| \le \sup_{x\in K}\left(\int_F+\int_{F^c}\right)|f(x-tz)-f(x)|\,|\phi(z)|\,dz$

$\le \varepsilon\|\phi\|_1 + 2\|f\|_u\int_{F^c}|\phi|\,dz \le \varepsilon + 2\varepsilon\|f\|_u.$ QED

Thm 8.15 Let $\phi \in L^1$ with $\int \phi\, dx = 1$. Suppose there exist $C>0$ and $\varepsilon>0$ such that $|\phi(x)| \le C(1+|x|)^{-n-\varepsilon}$ $\forall x \in \mathbb{R}^n$. If $f \in L^p$ $(1 \le p \le \infty)$ then $f * \phi_t(x) \longrightarrow f(x)$ as $t \to 0$ for every x in the Lebesgue set L_f of f.

Recall $\odot$ $L_f = \left\{ x \in \mathbb{R}^n : \lim\limits_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0 \right\}$.

$\odot$ Thm 3.20 $f \in L^1_{loc}(\mathbb{R}^n) \Longrightarrow m(L_f^c) = 0$.

Skip the Pf of Thm 8.15.

Prop. 8.17 C_c^∞ (and hence $\mathcal{S}$) is dense in L^p $(1 \le p < \infty)$ and in C_0.

Pf C_c is dense in L^p. Let $\phi \in C_c^\infty$, $\operatorname{supp}\phi = \overline{B(0,1)}$, $\phi \ge 0$, $\int \phi\, dx = 1$. Then $\forall g \in C_c$, $g * \phi_t \in C_c^\infty$, $g * \phi_t \to g$ in L^p.
The same argument applies if L^p is replaced by C_0 and $\|\cdot\|_p$ by $\|\cdot\|_u$. QED

<u>The C^∞ Urysohn Lemma</u> Let K be a compact subset and U an open subset of $\mathbb{R}^n$ such that $K \subseteq U$. Then there exists $f \in C_c^\infty(\mathbb{R}^n; [0,1])$ such $K \prec f \prec U$.

<u>Pf</u> Let $\delta = \mathrm{dist}(K, U^c) = \inf\{|x-y| : x \in K,\ y \in U^c\}$. Since K is compact, $\delta > 0$. Let $V = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) < \frac{\delta}{3}\}$. Choose $\phi \in C_c^\infty$, $\phi \geq 0$, $\mathrm{supp}\,\phi \subseteq \overline{B(0,1)}$, and $\int \phi\, dx = 1$. Set $f = \chi_V * \phi_{\delta/3}$. Then $f \in C_c^\infty(\mathbb{R}^n)$, $0 \leq f \leq 1$, $f = 1$ on K, and $\mathrm{supp}(f) \subseteq \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leq 2\delta/3\} \subseteq U$. <u>QED</u>