Wednesday, 4/29/2020, Lecture 14

— Finish §8.2 (see notes of last lecture)

— Review for the midterm exam

$\odot$ X: LCH. Some spaces / Properties.

$C_c(X) = \{$ all compactly supported continuous functions on $X\}$

$C_0(X) = \{$ all continuous functions on X that vanish at $\infty \}$

Prop. 4.35 $\quad \forall \varepsilon > 0$: $\{|f| \geq \varepsilon\}$ is compact in X.

$$C_c(X) \subseteq \overline{C_c(X)} = C_0(X) \subseteq BC(X) \subseteq C(X).$$

All the same if X is compact. $\quad$ Bound + Cont.

Prop. 4.31 X: LCH. K (compact) $\subseteq U$ (open) $\subseteq X$
$\Longrightarrow \exists$ precompact open V s.t. $K \subseteq V \subseteq \overline{V} \subseteq U$.

Urysohn's Lemma X: LCH. K (compact) $\subseteq U$ (open) $\subseteq X$ $\Longrightarrow \exists f \in C_c(X; [0,1])$ s.t. $K \prec f \prec U$ (i.e., $f = 1$ on K and $\operatorname{supp}(f) \subseteq U$).

Radon measures: Concepts / properties

Def. A Borel meas. μ on LCH space X is a Radon measure, if

- μ is finite on compact sets;
- μ is outer regular; and
- μ is inner regular on all open sets.

Be able to check if a Borel meas. is a Radon or not.

Some properties of a Radon measure μ on an LCH space X.

Prop. 7.5: $E \in \mathcal{B}_X$ is μ σ-finite $\Longrightarrow$ μ is regular on E.

σ-finite Radon meas. $\Longrightarrow$ regular.

X: σ-compact. μ: Radon on X $\Longrightarrow$ μ is regular.

The squeeze thm for a σ-finite Radon meas. μ

$\forall E \in \mathcal{B}_X$, $\forall \varepsilon > 0$, $\exists K$ compact, $\exists U$ open. s.t. $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \varepsilon$.

$\exists A$ (F_σ set) $\subseteq E \subseteq B$ (G_δ set) s.t. $\mu(B \setminus A) = 0$.

Thm 7.8: X: LCH. every open set is σ-compact. A Borel meas. is Radon $\Longleftrightarrow$ It is finite on compact sets.

μ: Radon $\Longrightarrow$ $C_c(X)$ is dense in $L^p(\mu)$ $(1 \le p < \infty)$.

Lusin's Thm.

① The Riesz Representation Thms X: LCH.

⊙ $I : C_c(X) \to \mathbb{C}$ linear + positive $\Longrightarrow \exists!$ Radon meas. μ on X s.t. $I(f) = \int f\,d\mu \ \forall f \in C_c(X)$.

Moreover,

U open: $\mu(U) = \sup\{ I(f) : f \in C_c(X, [0,1]),\ f \prec U \}$.

K: compact, $\mu(K) = \inf\{ I(f) : f \in C_c(X),\ f \geq \chi_K \}$.

⊙ $C_0(X)^* = M(X) = \{$all complex Radon measures$\}$

$\mu \in M(X)$: $I(f) = \int f\,d\mu \ \forall f \in C_0(X)$. $I \in C_0(X)^*$.

<u>Concepts</u> μ is a signed Radon measure $\Longleftrightarrow$ Jordan decomp: $\mu = \mu^+ - \mu^-$. $\mu^\pm$: Radon.

μ is a complex Radon meas. $\Longleftrightarrow$ $\mathrm{Re}\,\mu$, $\mathrm{Im}\,\mu$ are signed Radon meas. $\Longleftrightarrow$ $|\mu|$ is a Radon.

μ: Radon. $\phi \in L^1(\mu)$, $\nu(E) = \int_E |\phi| \, d\mu \quad \forall E \in \mathcal{B}_X$.

$\Longrightarrow$ ν is a <u>finite</u> Radon measure

$I : C_0(X) \to \mathbb{C}$ linear + positive $\Longrightarrow$ bounded.

<u>weak convergence</u> $f_n \to f$ in $C_0(X)$:

$\int f_n \, d\mu \to \int f \, d\mu \quad \forall \mu \in M(X)$.

$\Longleftrightarrow$ $\sup_{n \geq 1} \|f_n\| < \infty$ and $f_n \to f$ pointwise on X.

<u>weak-* convergence</u> (i.e., vague convergence)

$\mu_n, \mu \in M(X)$:

$\mu_n \to \mu$ vaguely: $\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in C_0(X)$.

$\Longrightarrow$ $\sup_{n \geq 1} \|\mu_n\| < \infty$.

$\mu_n \to \mu$ vaguely $\not\Longrightarrow$ $\mu_n(E) \to \mu(E) \quad \forall E \in \mathcal{B}_X$.

LSC and USC Functions

Def. $f : X \to (-\infty, \infty]$ is LSC if $\{f > a\}$ is open $\forall a \in \mathbb{R}$.

$f : X \to [-\infty, \infty)$ is USC if $\{f < a\}$ is open $\forall a \in \mathbb{R}$.

- U open $\Longrightarrow$ χ_U is LSC.

Each $g \in \mathcal{G}$ is LSC $\Longrightarrow$ $\sup\{g : g \in \mathcal{G}\}$ is LSC

X: LCH, $f \geq 0$ LSC on $X \Rightarrow f(x) = \sup\{g(x) : g \in C_c(X), 0 \leq g \leq f\}$.

- **Prop. 7.12**. $\mathcal{G}$: directed family of nonneg LSC functions on X (LCH), μ: Radon;

$$\int \sup\{g : g \in \mathcal{G}\}\, d\mu = \sup\left\{\int g\, d\mu : g \in \mathcal{G}\right\}.$$

- **Prop. 7.14** μ: Radon. $f \geq 0$: measurable:

$$\int f\, d\mu = \inf\left\{\int g\, d\mu : g \geq f,\ g : \text{LSC}\right\}$$

If $\{f \neq 0\}$ is σ-finite, then

$$\int f\, d\mu = \sup\left\{\int g\, d\mu : 0 \leq g \leq f,\ g : \text{USC}\right\}.$$

⊙ Product Radon Measures

X, Y: LCH, $(X, \mathcal{B}_X, \mu)$, $(Y, \mathcal{B}_Y, \nu)$: Radon meas. spaces

Thm 7.20 X, Y: 2nd countable $\Longrightarrow$ $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$ and $\mu\times\nu$ is a Radon meas. on $X\times Y$.

In general, $(X\times Y, \mathcal{B}_X\otimes\mathcal{B}_Y, \mu\times\nu)$ is a meas. space.

$f \longmapsto \int f\, d(\mu\times\nu)$ $\forall f\in C_c(X\times Y)$ is linear + positive

$\Rightarrow \exists!$ Radon $\mu\hat{\times}\nu$ on $X\times Y$ s.t. $\int f\, d(\mu\times\nu) = \int f\, d(\mu\hat{\times}\nu)$ $\forall f\in C_c$

Thm μ, ν: σ-finite Radon on X, Y, resp.

⊙ $\mu\hat{\times}\nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$ $\forall E\in\mathcal{B}_{X\times Y}$

⊙ $\mu\hat{\times}\nu = \mu\times\nu$ on $\mathcal{B}_X\otimes\mathcal{B}_Y$.

The Fubini–Tonelli Thm for Radon products.

$f\in L^1(\mu\hat{\times}\nu) \Longrightarrow \int f\, d(\mu\hat{\times}\nu) = \iint f\, d\mu\, d\nu = \iint f\, d\nu\, d\mu$.