Monday 5/4/2020, Lecture 15

§8.3 The Fourier Transform

- Trig functions $\cos kx$, $\sin kx$ $(k \in \mathbb{Z})$: relatively simple but can approx. other functions.
- Let $E_k(x) = e^{2\pi i k\cdot x}$. $\{E_k\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (n-dim torus).
- $\mathbb{T}^n \to \{(z_1, \dots, z_n) \in \mathbb{C}^n : \text{all } |z_j| = 1\}$: $(x_1, \dots, x_n) \mapsto (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$.
- $\mathbb{T}^n \leftrightarrow Q = [0,1]^n$: $m(\mathbb{T}^n) = m(Q) = 1$. $\mathbb{T}^n$: compact, Hausdorff.
- $f \in L^2(\mathbb{T}^n)$: $\hat{f}(k) = \langle f, E_k \rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i \xi\cdot x}\,dx$ $(k \in \mathbb{Z}^n)$

  $\sum_{k\in\mathbb{Z}^n} \hat{f}(k) E_k$

  $f \in L^1(\mathbb{R}^n)$: $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i k\cdot x}\,dx$ $(\xi \in \mathbb{R}^n)$
- Topics: Def., basic properties, inversion, the Riemann-Lebesgue Lemma, Hausdorff-Young ineq., Plancherel Thm, and Poisson summation.

<u>Thm 8.20</u> $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$.

$\cos(2\pi k t) + i\sin(2\pi k t)$
$\|$

<u>Proof</u> Orthonormality: $\int_0^1 e^{2\pi i k t}\, dt = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$

The Stone-Weierstrass Thm $\Longrightarrow$ $\operatorname{Span}\{E_k : k \in \mathbb{Z}^n\}$ is dense in $C(\mathbb{T}^n)$ in <u>the unif. norm</u>, hence, in the L^2-norm. So, $\overline{\operatorname{Span}\{E_k : k \in \mathbb{Z}^n\}} = L^2(\mathbb{T}^n)$.

If $f \in L^2(\mathbb{T}^n)$ and $\langle f, E_k \rangle = 0$ $\forall k \in \mathbb{Z}^n$, then $f = 0$ in $L^2(\mathbb{T}^n)$. Hence, the result is true by Thm 5.27. <u>QED</u>

<u>Def</u> If $f \in L^2(\mathbb{T}^n)$ then <u>the Fourier transform</u> $\hat{f}$, a function on $\mathbb{Z}^n$, is $\hat{f}(k) = \langle f, E_k \rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k x}\, dx$.

<u>The Fourier series</u> of f is $\sum_{k \in \mathbb{Z}^n} \hat{f}(k) E_k$, $\hat{f}(k)$ are <u>the Fourier coefficients</u>

The Hausdorff-Young ineq. $1 \le p \le 2$, q, p: conjugate.

$f \in L^p(\mathbb{T}^n) \Longrightarrow \hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \le \|f\|_p$.

Proof $p=1$. $q=\infty$. $\|\hat{f}\|_\infty \le \|f\|_1$. $p=2$. $q=2$. $\|\hat{f}\|_2 = \|f\|_2$

(Since $\{E_k\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis, $\sum_k |\hat{f}(k)|^2 = \|f\|_2^2$).

General case by the Riesz-Thorin interpolation Thm:

- $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$: meas. space, ν: σ-finite;
- $p_0, p_1, q_0, q_1 \in [0,\infty]$, $0<t<1$: $\dfrac{1}{p_t} = \dfrac{1-t}{p_0} + \dfrac{t}{p_1}$, $\dfrac{1}{q_t} = \dfrac{1-t}{q_0} + \dfrac{t}{q_1}$;
- $T: L^{p_0}(\mu) + L^{p_1}(\mu) \longrightarrow L^{q_0}(\nu) + L^{q_1}(\nu)$ linear.

$\|Tf\|_{q_0} \le M_0 \|f\|_{p_0}$ $\forall f \in L^{p_0}(\mu)$, $\|Tf\|_{q_1} \le M_1 \|f\|_{p_1}$ $\forall f \in L^{p_1}(\mu)$

$\Longrightarrow \|Tf\|_{q_t} \le M_0^{1-t} M_1^t \|f\|_{p_t}$ $\forall f \in L^{p_t}(\mu)$

Here, $Tf = \hat{f}$. $\dfrac{1}{p} = \dfrac{1-t}{1} + \dfrac{t}{2} = 1 - \dfrac{t}{2} \in (\tfrac{1}{2}, 1)$, $p \in (1,2)$

$\dfrac{1}{q} = \dfrac{1-t}{\infty} + \dfrac{t}{2} = \dfrac{t}{2} \in (0, 1/2)$, $q \in (2,\infty)$. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

$M_0 = M_1 = 1$. $\Longrightarrow \|Tf\|_q \le \|f\|_p$. QED

<u>Def.</u> <u>The Fourier transform</u> of $f \in L^1(\mathbb{R}^n)$ is $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \quad (\xi \in \mathbb{R}^n)$.

Note. $|\mathcal{F}f(\xi)| \leq \int_{\mathbb{R}^n} |f(x)| \, |e^{-2\pi i \xi \cdot x}| dx = \|f\|_1 \Longrightarrow \|\mathcal{F}f\|_u \leq \|f\|_1$

$\mathcal{F}f$ is continuous by the DCT. So, $\underline{\mathcal{F}: L^1(\mathbb{R}^n) \longrightarrow BC(\mathbb{R}^n)}$.

<u>Thm 8.22</u> Let $f, g \in L^1(\mathbb{R}^n)$.

(a) $\widehat{\tau_y f}(\xi) = e^{-2\pi i \xi \cdot y} \hat{f}(\xi)$,

$\tau_\eta(\hat{f}) = \hat{h}$ where $h(x) = e^{2\pi i \eta \cdot x} f(x)$

$x - y = z$

<u>Pf</u> $\widehat{\tau_y f}(\xi) = \int f(x-y) e^{-2\pi i \xi \cdot x} dx = \int f(z) e^{-2\pi i \xi \cdot (y+z)} dz$

$= e^{-2\pi i \xi \cdot y} \int f(z) e^{-2\pi i \xi \cdot z} dz = e^{-2\pi i \xi \cdot y} \hat{f}(\xi)$.

Similar for the other formula.

(b) $T:\mathbb{R}^n\to\mathbb{R}^n$ linear and invertible, $S=(T^{-1})^*$

$$\Longrightarrow \widehat{f\circ T} = |\det T|^{-1}\,\hat f\circ S.$$

$\xi\cdot T^{-1}x = S\xi\cdot x$

① T is a rotation or translation $\Longrightarrow \widehat{f\circ T}=\hat f\circ T$.

② $Tx = t^{-1}x\ (t>0) \Longrightarrow \widehat{f\circ T}(\xi) = t^n \hat f(t\xi)$, i.e., $\hat f_t(\xi)=\hat f(t\xi)$.

<u>pf</u> $\widehat{f\circ T}(\xi) = \int f(Tx)\,e^{-2\pi i\xi\cdot x}\,dx = |\det T|^{-1}\int f(x)\,e^{-2\pi i\xi\cdot T^{-1}x}\,dx$

$= |\det T|^{-1}\int f(x)\,e^{-2\pi i S\xi\cdot x}\,dx = |\det T|^{-1}\hat f(S\xi)$

$\int g = |\det T|\int g\circ T$

(c) $\widehat{f*g} = \hat f\,\hat g$. (Note: Young's ineq. $\Longrightarrow \|f*g\|_1\le\|f\|_1\|g\|_1$.)

<u>pf</u> $\widehat{f*g}(\xi) = \int\int f(x-y)\,g(y)\,e^{-2\pi i\xi\cdot x}\,dy\,dx$

$\overset{\text{Fubini}}{=} \iint f(x-y)\,e^{-2\pi i\xi\cdot(x-y)}\,g(y)\,e^{-2\pi i\xi\cdot y}\,dx\,dy$

$= \hat f(\xi)\int g(y)\,e^{-2\pi i\xi\cdot y}\,dy = \hat f(\xi)\,\hat g(\xi).$

(d) $x^\alpha f(x)\in L^1\ \forall|\alpha|\le k \Longrightarrow \hat f\in C^k$, $\partial^\alpha\hat f = \widehat{(-2\pi i x)^\alpha f}$.

<u>pf</u> $\partial^\alpha\hat f(\xi) = \partial_\xi^\alpha\int f(x)\,e^{-2\pi i\xi\cdot x}\,dx = \int f(x)\,(-2\pi i x)^\alpha\,e^{-2\pi i\xi\cdot x}\,dx.$

(e) $f \in C^k$, $\partial^\alpha f \in L^1 \ \forall |\alpha| \le k$, $\partial^\alpha f \in C_0 \ \forall |\alpha| \le k-1$

$$\Longrightarrow \widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$$

Integration by parts

Pf $n = |\alpha| = 1$: $\widehat{f'}(\xi) = \int f'(x) e^{-2\pi i \xi \cdot x} dx = f(x) e^{-2\pi i \xi \cdot x} \Big|_{-\infty}^{\infty}$

$- \int f(x)(-2\pi i \xi) e^{-2\pi i \xi \cdot x} dx = 2\pi i \xi \hat{f}(\xi)$.

General case: similar, induction. QED