Wednesday, 5/6/2020 Lecture 16

§8.3 The Fourier Transform (cont'd)

⊙ $f \in L^2(\mathbb{T}^n)$: The Fourier transform (FT) of f is $\hat{f}: \mathbb{Z}^n \to \mathbb{C}$: $\hat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x} dx$ $(\forall k \in \mathbb{Z}^n)$

The Fourier series is $\sum_k \hat{f}(k) E_k$, $E_k(x) = e^{2\pi i k\cdot x}$

The Hausdorff-Young ineq. $\|\hat{f}\|_{p'} \le \|f\|_p$ $(1 \le p \le 2)$.

⊙ $f \in L^1(\mathbb{R}^n)$. The FT of f is $\mathcal{F}f = \hat{f}$:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi\cdot x} dx \quad (\forall \xi \in \mathbb{R}^n).$$

⊙ $\mathcal{F}: L^1(\mathbb{R}^n) \longrightarrow BC(\mathbb{R}^n)$, linear.

[Later: $\mathcal{F}: L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$.]

⊙ $\widehat{f * g} = \hat{f}\hat{g}$

⊙ $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.

Prop. 8.24 (The FT of a Gaussian)

If $f(x)=e^{-\pi|x|^2}$ then $\hat{f}(\xi)=e^{-\pi|\xi|^2}$.

[If $a>0$ and $g(x)=e^{-\pi a^2|x|^2}$ then $\hat{g}(\xi)=a^{-n}e^{-\pi|\xi|^2/a^2}$.]

Pf First, assume $n=1$. $\hat{f}(\xi)=\int_{-\infty}^{\infty}e^{-\pi x^2}e^{-2\pi i\xi x}\,dx$.

$$\hat{f}'(\xi)=-2\pi i\int_{-\infty}^{\infty}x e^{-\pi x^2}e^{-2\pi i\xi x}\,dx$$

$$= i\int_{-\infty}^{\infty}\left(e^{-\pi x^2}\right)' e^{-2\pi i\xi x}\,dx$$

$$= i\, e^{-2\pi x^2}e^{-2\pi i\xi x}\Big|_{x=-\infty}^{x=\infty} - 2\pi\xi\int_{-\infty}^{\infty}e^{-\pi x^2}e^{-2\pi i\xi x}\,dx$$

$$= -2\pi\xi\,\hat{f}(\xi).$$

$$\frac{d}{d\xi}\left(e^{\pi\xi^2}\hat{f}(\xi)\right)=e^{\pi\xi^2}\left[\hat{f}'(\xi)+2\pi\xi f(\xi)\right]=0$$

$\hat{f}(\xi)=c_0e^{-\pi\xi^2}$. $\quad c_0=\hat{f}(0)=\int_{-\infty}^{\infty}e^{-\pi x}\,dx=1$.

Now, general $n\geq 2$: $\hat{f}(\xi)=\prod_{j=1}^{n}\int_{-\infty}^{\infty}e^{-\pi x_j^2}e^{-2\pi i\xi_j\cdot x_j}\,dx_j$

$$=\prod_{j=1}^{n}e^{-\pi\xi_j^2}=e^{-\pi|\xi|^2}.$$

QED

The <u>Riemann-Lebesgue Lemma</u> $\mathcal{F}(L^1(\mathbb{R}^n)) \subseteq C_0(\mathbb{R}^n)$.

$\forall f \in L^1(\mathbb{R}^n)$ $\quad \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx$. $\quad \hat{f} \in BC(\mathbb{R}^n)$

<u>Note</u>

- $\lim\limits_{\xi \to \infty} \int_{\mathbb{R}^n} f(x) \begin{matrix}\cos \\ \sin\end{matrix} (2\pi \xi x) dx = 0$.
- If $f \in L^2(\mathbb{T}^n)$, then $\|f\|_2^2 = \sum_k |\hat{f}(k)|^2 \Rightarrow \hat{f}(k) \to 0$

Parseval

$\Longrightarrow \begin{matrix}\cos 2\pi k x \\ \sin 2\pi k x\end{matrix} \longrightarrow 0$ weakly.

<u>Pf</u> If $f \in C^1 \cap C_c$ then $\widehat{\partial^\alpha f}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$ $(|\alpha| = 1)$ is bounded. Since $\hat{f} \in BC(\mathbb{R}^n)$, we thus have $\hat{f} \in C_0(\mathbb{R}^n)$.

$\forall g \in L^1(\mathbb{R}^n)$ Since $C^1 \cap C_c$ is dense in $L^1(\mathbb{R}^n)$, $\exists g_n \in C^1 \cap C_c$ s.t. $\|g_n - g\|_1 \to 0$. Now, each $\hat{g}_n \in C_0(\mathbb{R}^n)$. Moreover, $\|\hat{g}_n - \hat{g}\|_u \le \|g_n - g\|_1 \to 0$. Since $C_0(\mathbb{R}^n)$ is closed in the uniform norm, $\hat{g} \in C_0(\mathbb{R}^n)$. <u>QED</u>

The inverse Fourier transform (IFT)

<u>Def</u> Let $f \in L^1(\mathbb{R}^n)$. The IFT of f is

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi = \hat{f}(-x), \quad x \in \mathbb{R}^n.$$

<u>The Fourier Inversion Thm</u> If $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$ then $\exists\, f_0 \in C_0(\mathbb{R}^n)$ s.t. $f = f_0$ a.e. and $\check{\hat{f}} = \hat{\check{f}} = f_0$.

- Note. FT of $e^{-|x|}\chi_{(0,\infty)}$ is not in $L^1(\mathbb{R})$.
- Corollary. $f \in L^1$, $\hat{f} = 0 \Rightarrow f = 0$ a.e. So, $\mathcal{F}$ is injective.

<u>Lemma</u> $f, g \in L^1(\mathbb{R}^n) \Longrightarrow \int \hat{f}\, g = \int f\, \hat{g}$.

<u>Pf</u> $\int \hat{f}(x)\, g(x)\, dx = \int\int f(y)\, e^{-2\pi i x\cdot y}\, dy\, g(x)\, dx$

$= \int\int f(y)\, g(x)\, e^{-2\pi i x\cdot y}\, dx\, dy = \int f(x)\, \hat{g}(x)\, dx.$ <u>QED</u>

Pf of the inversion thm. Let $t>0$ and $x\in\mathbb{R}^n$.

Def. $\phi_t(\xi)=e^{-\pi t^2|\xi|^2+2\pi i\xi\cdot x}$. Then $\hat{\phi}_t(y)=t^{-n}e^{-\pi|x-y|^2/t^2}$

$=g_t(x-y)$, where $g(x)=e^{-\pi|x|^2}$, $g_t(x)=t^{-n}g(x/t)$.

By the lemma,

$$\int\hat{f}(\xi)\,\phi(\xi)\,d\xi=\int f(\xi)\,\hat{\phi}(\xi)\,d\xi=\int f(\xi)\,g_t(x-\xi)\,d\xi=f*g_t(x).$$

Since $\int g(x)\,dx=1$, $f*g_t\longrightarrow f$ as $t\to 0$. Since $\hat{f}\in L^1$,

$$\int\hat{f}(\xi)\,\phi(\xi)\,d\xi=\int e^{-\pi t^2|\xi|^2+2\pi i\xi\cdot x}\hat{f}(\xi)\,d\xi\xrightarrow[t\to 0]{DCT}\int\hat{f}(\xi)\,e^{2\pi i\xi\cdot x}\,d\xi=\check{\hat{f}}(x)$$

Thus, $f=\check{\hat{f}}$ a.e. Similarly, $f=\hat{\check{f}}$ a.e. Let $f_0=\check{\hat{f}}$ $\in C_0(\mathbb{R}^n)$. Then, $\hat{\check{f}}=f_0$, $\check{\hat{f}}=f_0$, and $f=f_0$ a.e. QED