Friday, 5/8/2020. Lecture 17

§8.3 The Fourier Transform (cont'd)

- $f \in L^2(\mathbb{T}^n)$: $\hat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x} dx \quad (k \in \mathbb{Z}^n)$.

  $E_k(x) = e^{2\pi i k\cdot x}$: orthonormal. Parseval: $\|f\|^2_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2$.
- $f \in L^1(\mathbb{R}^n)$: $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi\cdot x} dx \quad (\xi \in \mathbb{R}^n)$.
  - $\mathcal{F}: L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$. 1-1, not onto. $\|\hat{f}\|_u \le \|f\|_{L^1}$.
  - $\widehat{f*g} = \hat{f}\hat{g}$, $\widehat{\partial^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$, $g(x) = e^{-\pi|x|^2} \Longrightarrow \hat{g} = g$.
  - The Riemann-Lebesgue Lemma: $f \in L^1(\mathbb{R}^n) \Longrightarrow \hat{f} \in C_0(\mathbb{R}^n)$.
  - The Fourier Inversion: $f \in L^1$, $\hat{f} \in L^1 \Longrightarrow \check{\hat{f}} = \hat{\check{f}} = f$ a.e.

Today:
- The FT of $f \in \mathcal{S}$
- The Plancherel Thm
- The Hausdorff-Young ineq.
- The Poisson Summation

Thm $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ is a homeomorphism.

Pf. Step 1: $f \in \mathcal{S} \Rightarrow \mathcal{F}f = \hat{f} \in \mathcal{S}$.

$\forall \alpha, \beta$: $x^\alpha \partial^\beta f \in L^1 \cap C_0$. Thm 8.22 (d)(e) $\Rightarrow \hat{f} \in C^\infty$ and

$$\widehat{x^\alpha \partial^\beta f} = (-1)^{|\alpha|} (2\pi i)^{|\beta| - |\alpha|} \partial^\alpha (\xi^\beta \hat{f}).$$

Thus $\partial^\alpha(\xi^\beta \hat{f})$ is bounded. Prop. 8.3 $\Rightarrow \hat{f} \in \mathcal{S}$.

Step 2: $\mathcal{F}$ is cont. on $\mathcal{S}$

Continuing, since $\int (1+|x|)^{-n-1} dx < \infty$, we have

$$\|\widehat{x^\alpha \partial^\beta f}\|_u \le \|x^\alpha \partial^\beta f\|_1 \le C \|(1+|x|)^{n+1} x^\alpha \partial^\beta f\|_u$$

for some $C > 0$ indep on f. Thus,

$$\|\hat{f}\|_{(N,\beta)} \le C_{N,\beta} \sum_{|\gamma| \le |\beta|} \|f\|_{(N+n+1,\gamma)}$$

cf. Pf of Prop. 8.3. Hence $\mathcal{F}$ is cont. on $\mathcal{S}$.

Step 3 Wrap up. $f \to \check{f}$ is also cont. since $\check{f}(x) = \hat{f}(-x)$. So $\mathcal{F}^{-1}$ is cont. But $\check{\hat{f}} = f$. So, onto, 1-1. QED

The Plancherel Thm $f \in L^1 \cap L^2 \Rightarrow \hat{f} \in L^2$.

$\mathcal{F}|_{L^1 \cap L^2}$ extends uniquely to a unitary isomorphism on $L^2(\mathbb{R}^n)$.

Pf Let $X = \{f \in L^1 : \hat{f} \in L^1\}$.

- $f \in X \Rightarrow f = \check{\hat{f}} \in L^\infty \Longrightarrow f \in L^2 \Longrightarrow X \subseteq L^2$.
- $\mathcal{S} \subseteq X$. $\mathcal{S}$ is dense in $L^2 \Longrightarrow X$ is dense in L^2.
- $\mathcal{F}$ preserves the L^2-inner product on X:

$\forall f, g \in X \subseteq L^2 : \langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$. i.e., $\int \hat{f}\,\overline{\hat{g}} = \int f\,\overline{g}$.

Let $h = \overline{\hat{g}}$. $\hat{h}(\xi) = \int \overline{\hat{g}(x)}\, e^{-2\pi i \xi \cdot x} dx = \int \overline{\hat{g}(x) e^{2\pi i \xi \cdot x}} dx$

$= \overline{\int \hat{g}(x) e^{2\pi i \xi \cdot x} dx} = \overline{\check{\hat{g}}(\xi)} = \overline{g(\xi)}$. Thus

$\int f \overline{g} = \int f \hat{h} \overset{\text{lemma 8.25}}{=\!=} \int \hat{f} h = \int \hat{f}\, \overline{\hat{g}}$.

- $\mathcal{F}(X) = X$ by the Fourier inversion thm.

Thus, $\mathcal{F}|_X$ extends uniquely to a unitary isomorphism on $L^2(\mathbb{R}^n)$ — linear, bijective, cont., and the inverse is cont.

Show that the extension agrees with $\mathcal{F}$ on $L^1 \cap L^2$: $\boxed{90}$

Let $f \in L^1 \cap L^2$, $g(x) = e^{-\pi |x|^2}$, $g_t(x) = t^{-n} g(x/t)$ $(t > 0)$.

Then $f * g_t \in L^1$ by Hölder's. $\widehat{f * g_t}(\xi) = \hat{f}(\xi)\, \widehat{g_t}(\xi) = \hat{f}(\xi) e^{-\pi t^2 |\xi|^2}$.

$\hat{f}$ is bounded. So, $\widehat{f * g_t} \in L^1$. Hence $f * g_t \in X$.

Now, $f * g_t \to f$ in L^1 and in L^2, as $f \in L^1 \cap L^2$. Hence

$\widehat{f * g_t} \to \hat{f}$ uniformly and in L^2, as $\|\hat{\cdot}\|_u \le \|\cdot\|_1$ and

$\mathcal{F}$ preserves the L^2-norm on X. QED

The Hausdorff-Young ineq.

$1 \le p \le 2$, $p' = p/(p-1)$.

$f \in L^p(\mathbb{R}^n) \Longrightarrow \hat{f} \in L^{p'}(\mathbb{R}^n)$ and $\|\hat{f}\|_{p'} \le \|f\|_p$.

Pf. By the Riesz-Thorin Thm and $\|\hat{f}\|_\infty \le \|f\|_1$

and $\|\hat{f}\|_2 = \|f\|_2$. QED

Question $f \in L^1(\mathbb{R}^n)$. Construct $\tilde{f} \in L^2(\mathbb{T}^n)$ from f.

① $\tilde{f}(x) = \sum_{k \in \mathbb{Z}^n} f(x-k)$.

② $\hat{f}(\xi)$. $\tilde{f}(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x}$

$\Big\} \Longrightarrow$ same $\tilde{f}$?

Thm 8.31 Let $f \in L^1(\mathbb{R}^n)$. The series $\sum_{k \in \mathbb{Z}^n} \tau_k f$ converges a.e. and in $L^1(\mathbb{T}^n)$ to some $Pf \in L^1(\mathbb{R}^n)$ s.t. $\|Pf\|_1 \leq \|f\|_1$. Moreover, $\widehat{Pf}(k) = \hat{f}(k)$ $\forall k \in \mathbb{Z}^n$.

Pf Let $A = [-\frac{1}{2}, \frac{1}{2})^n$. $\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}^n} (k + A)$, disjoint.

$$\int_A \sum_k |\tau_k f(x)| dx = \sum_k \int_A |f(x-k)| dx = \sum_k \int_{k+A} |f(x)| dx = \int_{\mathbb{R}^n} |f(x)| dx.$$

Hence, $\sum_k \tau_k f \to Pf$ a.e. and in L^1 for some $Pf \in L^1$, and $\|Pf\|_1 \leq \|f\|_1$. Moreover, $\widehat{Pf}(k) = \int_A \sum_m f(x-m) e^{-2\pi i k \cdot x} dx$

$$= \sum_m \int_{A+m} f(x) e^{-2\pi i k \cdot (x+m)} dx = \sum_m \int_{A+m} f(x) e^{-2\pi i k \cdot x} dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i k x} dx$$

$= \hat{f}(k)$. QED

The Poisson Summation Formula. Let $f \in C(\mathbb{R}^n)$. Suppose $|f(x)| \leq C(1+|x|)^{-n-\varepsilon}$ and $|\hat{f}(\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$ for some $C>0$, $\varepsilon>0$. Then,

$$\sum_{k \in \mathbb{Z}^n} f(x+k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x},$$

both converging absolutely and uniformly on $\mathbb{T}^n$. In particular,

$$\sum_{k \in \mathbb{Z}^n} f(k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k).$$

Pf Clearly, $f \in L^1$ and $\hat{f} \in L^1$. Moreover,

$$\int (1+|x|)^{-n-\varepsilon} dx < \infty \Longrightarrow \sum_k (1+|k|)^{-n-\varepsilon} < \infty \Longrightarrow$$

$\sum_k \hat{f}(k) e^{2\pi i k\cdot x}$ converges absolutely and uniformly on $\mathbb{T}^n$. Let $x \in [0,1]^n$. Then $|x| \le \sqrt{n}$. Let $k \in \mathbb{Z}^n$. Since $(1+|x|)^{-n-\varepsilon} \le (2\sqrt{n})^{n+\varepsilon} (2\sqrt{n}+|x|)^{-n-\varepsilon}$, we have

$|f(x+k)| \le C(2\sqrt{n})^{n+\varepsilon}(2\sqrt{n}+|x+k|)^{-n-\varepsilon} \le C(2\sqrt{n})^{n+\varepsilon}(2\sqrt{n}+|k|-|x|)^{-n-\varepsilon}$
$\le C(2\sqrt{n})^{n+\varepsilon}(\sqrt{n}+|k|)^{-n-\varepsilon}$. Hence, $\sum_{k\in\mathbb{Z}^n} f(x+k)$ converges absolutely and uniformly on $\mathbb{T}^n$. Let $Pf(x) = \sum_k f(x+k) = \sum_k \tau_{-k} f(x)$. Then, $Pf \in C(\mathbb{T}^n) \subseteq L^2(\mathbb{T}^n)$. By Thm 8.31, $\sum_k \hat{f}(k) e^{2\pi i k\cdot x} = \sum_k \widehat{Pf}(k) e^{2\pi i k\cdot x}$ which converges to Pf in L^2. Since this series also converges uniformly, we have $\sum_k \hat{f}(k) e^{2\pi i k\cdot x} = Pf(x)$ $\forall x \in \mathbb{R}^n$. QED