Monday, 5/11/2020. Lecture 18

§8.4 Summation of Fourier Integrals / Series.

Approximation of f by f^t as $t \to 0$

$f^t = \widecheck{\hat{f}\,\Phi^t}$, $\Phi^t(\xi) = \Phi(t\xi)$, $\Phi \in L^1 \cap C_0$, $\check{\Phi} \in L^1$, $\Phi(0) = 1$.

The kernel $\phi(x) = \check{\Phi}(x)$. $\boxed{\int \phi = \int \check{\Phi}(\xi) e^{-2\pi i \xi \cdot 0} d\xi = \hat{\check{\Phi}}(0) = \Phi(0) = 1}$

- Gauss: $\phi(x) = \check{\Phi}(x) = e^{-\pi |x|^2}$, $\Phi(\xi) = e^{-\pi |\xi|^2}$.
- Poisson: $\phi(x) = \check{\Phi}(x)$. $\Phi(\xi) = e^{-2\pi |\xi|}$

  $n=1$: $\phi(x) = \int_{-\infty}^0 e^{2\pi(1+ix)\xi} d\xi + \int_0^\infty e^{2\pi(1-ix)\xi} d\xi = \frac{1}{\pi(1+x^2)}$.

  $n \geq 1$: $\phi(x) = \frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\pi^{(n+1)/2}} \cdot \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$.

  $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ $(\operatorname{Re} z > 0)$. $\Gamma(n+1) = n!$

- $\Phi(\xi) = \max(1-|\xi|, 0)$. $n=1$: $\phi(x) = \check{\Phi}(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$.

Thm 8.35 Let $\Phi \in L^1 \cap C_0$, $\Phi(0)=1$, $\phi := \check{\Phi} \in L^1$.
Let $f \in L^1 + L^2$. For $t>0$, set

$$f^t(x) = \int \hat{f}(\xi)\,\Phi(t\xi)\,e^{2\pi i \xi\cdot x}\,d\xi = \left(\hat{f}\,\Phi(t\cdot)\right)^{\vee}, \quad x \in \mathbb{R}^n.$$

Then $\int \phi = 1$, $f^t = f * \phi_t$ where $\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right)$.

In particular,

(1) $f \in L^p\ (1 \le p < \infty) \Longrightarrow f^t \in L^p$, $\|f^t - f\|_p \to 0$ as $t \to 0$.

(2) f is bounded + unif. cont. $\Longrightarrow f^t$ is bounded + unif. cont. and $f^t \to f$ unif. as $t \to 0$.

(3) Suppose $|\phi(x)| \le C(1+|x|)^{-n-\varepsilon}$ for some $C, \varepsilon > 0$. Then $f^t(x) \to f(x)$ as $t \to 0$ for any x in the Lebesgue set of f.

Remark. Once $f^t = f * \phi_t$ is proven, the parts (1)–(3) follow immediately from the previous results; cf. Thm 8.14 and Thm 8.15.

<u>Lemma</u> $f, g \in L^2 \Rightarrow (\hat{f}\hat{g})^\vee = f * g$.

<u>Pf</u> $\hat{f}\hat{g} \in L^1$ by Plancerel and Hölder's. So, $(\hat{f}\hat{g})^\vee$ exists. $\forall x \in \mathbb{R}^n$. Let $h(y) = \overline{g(x-y)}$. Then, we have $\hat{h}(\xi) = \overline{\hat{g}(\xi)} e^{-2\pi i \xi \cdot x}$. Since $\mathcal{F}$ is unitary on L^2,

$f*g(x) = \int f \bar{h} = \int \hat{f} \bar{\hat{h}} = \int \hat{f}(\xi)\hat{g}(\xi) e^{2\pi i \xi\cdot x} d\xi = (\hat{f}\hat{g})^\vee(x)$. <u>QED</u>

<u>Pf of Thm</u> $f \in L^1 + L^2 \Rightarrow f = f_1 + f_2$, $f_1 \in L^1$, $f_2 \in L^2$. So, $\hat{f_1} \in L^\infty$, $\hat{f_2} \in L^2$. Now, $\Phi \in L^1 \cap C_0 \subseteq L^2$. So, the integral defining f^t converges absolutely. Since $\phi = \check{\Phi}$, $\hat{\phi} = \hat{\check{\Phi}} = \Phi$, and $\Phi(t\xi) = \widehat{\phi_t}(\xi)$. Now, $\phi, \Phi \in L^1 \Rightarrow f_1 * \phi \in L^1$, $\hat{f_1}\hat{\phi} = \hat{f_1}\Phi \in L^1$. Thus, by the Fourier inversion thm, we have $f_1^t(x)$

$= \int \hat{f_1}(\xi)\Phi(t\xi)e^{2\pi i \xi\cdot x} d\xi = \int \hat{f_1}(\xi)\widehat{\phi_t}(\xi) e^{2\pi i \xi\cdot x} d\xi$

$= \int \widehat{f_1 * \phi_t}(\xi) e^{2\pi i \xi\cdot x} dx = f_1 * \phi_t(x)$. Now, by the Plancerel Thm, $\phi = \check{\Phi} \in L^2$. By the Lemma, $f_2^t(x) =$

$\int \hat{f_2}(\xi)\Phi(t\xi) e^{2\pi i \xi\cdot x} d\xi = (\hat{f_2}\widehat{\phi_t})^\vee(x) = f_2 * \phi_t(x)$. <u>QED</u>

Thm 8.36 Let $\Phi \in C(\mathbb{R}^n)$ be such that $|\Phi(\xi)| \le C(1+|\xi|)^{-n-\varepsilon}$, $|\check{\Phi}(x)| \le C(1+|x|)^{-n-\varepsilon}$, and $\Phi(0)=1$, where $C>0$, $\varepsilon>0$. For $f \in L^1(\mathbb{T}^n)$ and $t>0$, def. $f^t(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\Phi(tk)e^{2\pi i k\cdot x}$.

(1) $f \in L^p(\mathbb{T}^n)$ $(1\le p<\infty) \Longrightarrow \|f^t - f\|_p \to 0$ as $t \to 0$.

$f \in C(\mathbb{T}^n) \Longrightarrow f^t \to f$ unif. as $t \to 0$.

(2) $f^t(x) \to f(x)$ $\forall x$ in the Lebesgue set of f.

Pf (May skip in class) (1) Let $\phi = \check{\Phi}$, $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$. Then $\widehat{\phi_t}(\xi) = \Phi(t\xi)$. Poisson sum. $\Longrightarrow \sum_{k\in\mathbb{Z}^n} \phi_t(x-k) = \sum_{k\in\mathbb{Z}^n} \Phi(tk)e^{2\pi i k\cdot x} =: \psi_t(x)$.

So, $\widehat{f*\psi_t}(k) = \hat{f}(k)\,\widehat{\psi_t}(k) = \hat{f}(k)\Phi(tk) = \widehat{f^t}(k) \Rightarrow f^t = f*\psi_t$.

Young's ineq. + Thm 8.31 $\Longrightarrow \|f^t\|_p \le \|f\|_p\|\psi_t\|_1 \le \|f\|_p\|\phi_t\|_1 = \|f\|_p\|\phi\|_1$.

$\Longrightarrow f \mapsto f^t$ are unif. bounded on L^p, $1\le p\le\infty$.

Since Φ is cont. $\Phi(0)=1$; $f^t \to f$ unif. (and hence in $L^p(\mathbb{T}^n)$), if f is a trig. function (i.e., $\hat{f}(k)=0$ except finitely many k). Stone-Weierstrass: trig. functions are dense in $C(\mathbb{T}^n)$, hence in $L^p(\mathbb{T}^n)$ $(1 \le p < \infty)$. Hence, (1) is true (cf. Prop. 5.17).

(2) Let $x \in L_f$ (Lebesgue set of f). By translation, we may assume $x=0$. Set $Q=[-\frac{1}{2},\frac{1}{2})^n$. Then $f^t(0)$

$$= f * \psi_t(0) = \int_Q f(x)\psi_t(-x)\,dx = \int_Q f(x)\,\phi_t(-x)dx + \sum_{k\neq 0}\int_Q f(x)\phi_t(-x+k)\,dx.$$

Since $|\phi_t(x)| \le C\,t^{-n}(1+t^{-1}|x|)^{-n-\varepsilon} \le C\,t^{\varepsilon}|x|^{-n-\varepsilon}$ $(x \in Q,\ k \neq 0)$, we have $|\phi_t(-x+k)| \le C\,2^{n+\varepsilon}t^{\varepsilon}|k|^{-n-\varepsilon}$, hence,

$$\sum_{k\neq 0}\int_Q |f(x)\phi_t(-x+k)|dx \le \Big[C\,2^{n+\varepsilon}\|f\|_1 \sum_{k\neq 0}|k|^{-n-\varepsilon}\Big]t^{\varepsilon} \longrightarrow 0 \text{ as } t\to 0.$$

Def. $g = f\chi_Q \in L^1(\mathbb{R}^n)$. Then $0 \in L_Q$ (Lebesgue point). So, by Thm 8.15,

$$\lim_{t\to 0}\int_Q f(x)\,\phi_t(-x)dx = \lim_{t\to 0} g*\phi_t(0) = g(0) = f(0).$$

QED