Wednesday, 5/13/2020. Lecture 19

§8.5 Pointwise Convergence of Fourier Series (for one-variable functions)

Let $f \in L^1(\mathbb{T})$. Def. the Fourier series of f:

$$f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}, \quad \hat{f}(k) = \int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx = \int_0^1 \cdots = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots.$$

$\hat{f}(k)$: Fourier coefficients.

Def. the mth symmetric partial sum:

$$S_m f(x) = \sum_{k=-m}^{m} \hat{f}(k) e^{2\pi i k x} \quad \text{(a trig. polynomial)}$$

Question: • Under what conditions $S_m f(x) \to f(x)$, i.e., the Fourier series converges to f at x? (pointwise convergence)

• $S_m f \to f$ in L^p?

$$S_m f(x) = \sum_{-m}^{m} \int_0^1 f(y) e^{-2\pi i k y} dy \, e^{2\pi i k x}$$

$$= \int_0^1 f(y) \sum_{-m}^{m} e^{2\pi i k (x-y)} dy = f * D_m(x)$$

$D_m(x) = \sum_{-m}^{m} e^{2\pi i k x}$: the Dirichlet kernel

$$D_m(x) = e^{-2\pi i m x} \sum_{0}^{m} e^{2\pi i k x} = e^{-2\pi i m x} \frac{e^{2\pi i (2m+1)x} - 1}{e^{2\pi i x} - 1}$$

$$= \frac{e^{(2m+1)\pi i x} - e^{-(2m+1)\pi i x}}{e^{\pi i x} - e^{-\pi i x}} = \frac{\sin (2m+1)\pi x}{\sin \pi x}.$$

- D_m is real-valued, even function.
- $D_m(0) = 2m+1$ $\quad \left(D_m(0) = \lim_{x \to 0} D_m(x) \right)$.
- Known: the Lebesgue const. $\|D_m\|_1 \to \infty$.

In fact, $\|D_m\|_1 \geq 4/\pi \sum_{j=1}^{m} \frac{1}{j}$. So, $\|D_m\| = O(\ln m)$.
(See HW#6)

Main Thm (Thm 8.43). If $f \in BV(\mathbb{T})$ then

$$\lim_{m\to\infty} S_m f(x) = \frac{1}{2}\left[f(x-) + f(x+)\right] \quad \forall x \in \mathbb{R}.$$

Lemma 8.41 Let $\phi, \psi : [a,b] \to \mathbb{R}$ be s.t. ϕ is monotone and right-cont, $\psi \in C([a,b])$. Then $\exists\, \eta \in [a,b]$ s.t.

$$\int_a^b \phi(x)\psi(x)\,dx = \phi(a)\int_a^\eta \psi(x)\,dx + \phi(b)\int_\eta^b \psi(x)\,dx.$$

Pf Assume $\phi(a)=0$ (otherwise consider $\phi - \phi(0)$) and ϕ is increasing (otherwise consider $-\phi$). Let $\Psi(x) = \int_x^b \psi(t)\,dt$. So, $\Psi' = -\psi$. By the integration by parts (cf. Thm 3.36) and $\phi(0)=0$, $\Psi(b)=0$, we get

$$\int_a^b \phi\psi = -\phi\Psi\Big|_{a=0}^{b=0} + \int_{(a,b)} \Psi(x)\,d\phi(x) = \int_{(a,b]} \Psi(x)\,d\phi(x).$$

Since ϕ increasing $\int_{(a,b)} d\phi = \phi(b) - \phi(a) = \phi(b)$. Assume $\begin{matrix} m \\ M \end{matrix} = \begin{matrix} \min \\ \max \end{matrix}\ \Psi$ over $[a,b]$. So, $m\,\phi(b) \le \int_{(a,b]} \Psi(x)\,d\phi(x) \le M\,\phi(b)$.

Hence, by the intermediate value thm, $\exists\, \eta \in [a,b]$. s.t.

$$\int_{(a,b]} \Psi(x)\,d\phi(x) = \Psi(\eta)\phi(b) = \phi(b)\int_\eta^b \psi(x)\,dx. \qquad \underline{\text{QED}}$$

Lemma 8.42 $\exists C > 0$ s.t. $\forall m \geq 0$, $\forall [a,b] \subseteq [-\frac{1}{2}, \frac{1}{2}]$.

$\left|\int_a^b D_m(x)dx\right| \leq C$. Moreover $\int_{-\frac{1}{2}}^{0} D_m(x)dx = \int_0^{\frac{1}{2}} D_m(x)dx = \frac{1}{2}$.

Pf D_m: even. $\int_{-\frac{1}{2}}^{0} D_m(x)dx = \int_0^{\frac{1}{2}} D_m(x) = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} D_m(x)\,dx$

$$= \frac{1}{2}\sum_{-m}^{m} \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i k x}dx}_{=0 \text{ if } k \neq 0.} = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dx = \frac{1}{2}.$$

$|x| \ll 1 \Rightarrow \dfrac{\pi x - \sin \pi x}{\pi x \sin \pi x} = O(x)$

For general $[a,b]$,

$$\int_a^b D_m(x)\,dx = \int_a^b \frac{\sin(2m+1)\pi x}{\pi x}dx + \underbrace{\int_a^b \sin(2m+1)\pi x \overbrace{\left(\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right)}\,dx}_{\text{bounded by a const.}}$$

Chang variable $y = (2m+1)\pi x$.

$$\int_a^b \frac{\sin(2m+1)\pi x}{\pi x}dx = \int_{(2m+1)\pi a}^{(2m+1)\pi b} \frac{\sin y}{\pi y}dy$$

$$= \frac{Si[(2m+1)\pi b] - Si[(2m+1)\pi a]}{\pi}$$ where $Si(x) = \int_0^x \frac{\sin y}{y}dy$.

$Si(x)$ is cont. $Si(x) \longrightarrow \pm\pi/2$ as $x \to \pm\infty$. So, $Si(x)$ is bounded. Thus, the first integral is also bounded by a const. indep. on a, b. QED