Monday, 3/30, Lecture 1

- Class web page: HW, exams, etc.
  - HW latex file / PDF on the web.
  - Midterm + Final exams: more details later.
- HW#1, due noon, Monday, 4/6.

This quarter:

ch7. Radon Measures

ch8. Elements of Fourier Analysis

ch9. Elements of Distribution Theory.

Ch7. Radon Measures

- Generalization of Lebesgue measure on $\mathbb{R}^n$ to LCH spaces
- Approach: measures $\longleftrightarrow$ linear functionals

$$F(f) = \int f\,d\mu \qquad F \longleftrightarrow \mu$$

Recall: $F \in [L^p(\mu)]^* \Longrightarrow \nu(E) = F(\chi_E)$ defines a signed measure, $\nu \ll \mu \Longrightarrow d\nu = g\,d\mu$.

- Topics:
  - The Riesz Representation Thm. Or: Positive Linear functionals on $C_c(X)$ (defining Radon measures)
  - Properties of Radon measures (e.g. Regularity, approximations, ...)
  - The dual of $C_0(X)$
  - Product Radon Measures

Review / Preparation

- X: LCH = $\underbrace{\text{Locally compact}}_{\forall x\in X\ \exists \text{ compact n.b.h. } N_x \text{ of } x}$ $\underbrace{\text{Hausdorff}}_{\forall x\neq y,\ \exists \text{ open, disjoint } U,V:\ x\in U,\ y\in V}$.

- **Urysohn's Lemma**

X: LCH. K (compact) $\subseteq U$ (open) $\subseteq X$.

Then, $\exists f\in C_c(X,[0,1])$ s.t. $f=1$ on K, $\mathrm{supp}(f)\subseteq U$.

Notation: $K \prec f \prec U$.

$f\prec U \iff f\in C_c(X,[0,1])$ and $\mathrm{supp}(f)\subseteq U$.

Recall $\mathrm{supp}(\varphi)$ = the closure of $\{x\in X: \varphi(x)\neq 0\}$.

- **Borel measures** on a LCH space X.

$\mathcal{B}_X$ = the Borel σ-algebra of X. It's the σ-alg. generated by all the open subsets of X.

$(X, \mathcal{B}_X)$: measurable space.

A Borel measure on X is a measure (nonnegative) on $\mathcal{B}_X$ or $(X, \mathcal{B}_X)$.

Let μ be a Borel measure on X.

- μ is inner regular at $E \in \mathcal{B}_X$, if
  $\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K : \text{compact}\}$.
- μ is outer regular at $E \in \mathcal{B}_X$, if
  $\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U : \text{open}\}$
- regular = inner + outer regular.
- μ is (inner, outer) regular if it is at each $E \in \mathcal{B}_X$.

§7.1 Positive Linear Functionals on $C_c(X)$

X: LCH. $C_c(X) = \{$all compactly supported continuous functions $f: X \to \mathbb{C}\}$.

Recall: $\overline{C_c(X)} = C_0(X) \subseteq BC(X) \subseteq B(X)$

norm $\|\cdot\| = \|\cdot\|_u = \|\cdot\|_\infty$

$C_0(X)$: cont. $\underbrace{f(\infty) = 0}_{\forall \varepsilon > 0 \ \{|f| \geq \varepsilon\} \text{ is compact.}}$

$BC(X)$: Bounded + cont.

$B(X)$: Bounded

Def $I: C_c(X) \to \mathbb{C}$ is <u>positive</u> if
$$\forall f \in C_c(X), \ f \geq 0 \Longrightarrow I(f) \geq 0.$$

Question: Do you have some examples of positive linear functionals?

Some nice property of such functionals

<u>Prop</u>. X: LCH. $I: C_c(X) \to \mathbb{C}$ positive + linear.
Then: $\forall K$: compact, $K \subseteq X$. $\exists\, C_K \geq 0$ s.t.
$f \in C_c(X)$, $\mathrm{supp}(f) \subseteq K \Longrightarrow |I(f)| \leq C_K \|f\|$.

<u>Note</u> I is not necessary bounded!
(This is Prob. #2 of HW#1.)

<u>Def</u>. X: LCH. A Borel measure on X is a <u>Radon measure</u> if it is

- finite on compact sets;
- outer regular; and
- inner regular at any open set.

(More general (or weaker) than regular)

Remarks

- If X is also σ-compact, then Radon = finite on compact sets + regular
- On $\mathbb{R}^n$: Radon = finite on compact sets. (Cf. Prob. #8. HW #1).

The Riesz Representation Theorem X: LCH. If $I : C_c(X) \to \mathbb{C}$ is positive + linear, then $\exists !$ Radon measure μ on X such that $I(f) = \int f \, d\mu$ $\forall f \in C_c(X)$.

Moreover

$\forall U (\text{open}) \subseteq X$: $\mu(U) = \sup\{ I(f) : f \in C_c(X),\ f \prec U \}$.

$\forall K (\text{compact}) \subseteq X$: $\mu(K) = \inf\{ I(f) : f \in C_c(X),\ f \geq \chi_K \}$.

A different way of constructing measures! (But we still need help from Caratheodory!)