Friday, 5/15/2020. Lecture 20

§8.5
- Pointwise convergence thm
- Localization thm
- Gibbs pheonomenon

Thm 8.43 $f \in BV(\mathbb{T}) \Rightarrow \lim_{m\to\infty} S_m f(x) = \frac{1}{2}[f(x-)+f(x+)]$ $\forall x \in \mathbb{R}$.

Pf Assume WLOG

- $x=0$. (otherwise: $\tau_{-x} f$).
- f is real-valued (otherwise consider $\operatorname{Re} f$, $\operatorname{Im} f$).
- f is right-cont (as replacing $f(t)$ by $f(t+)$ does not affect $S_m f(0)$ and $\frac{1}{2}[f(0+)+f(0-)]$.)

$f \in BV([-\frac{1}{2}, \frac{1}{2})) \Longrightarrow f = g - h$, g, h: increasing, right-cont. on $[-\frac{1}{2}, \frac{1}{2})$. Extend g, h 1-periodically on $\mathbb{R}$. $g, h \in BV(\mathbb{R})$.
Need only to consider one of g, h.

So, assume $x=0$, f is right-cont., increasing on $[-\frac{1}{2}, \frac{1}{2})$.

Since D_m is even,

$$S_m f(0) = f * D_m(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_m(0-x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) D_m(x)\,dx.$$

By Lemma 8.42, $\int_0^{\frac{1}{2}} D_m(x)\,dx = \int_{-\frac{1}{2}}^{0} D_m(x)\,dx = \frac{1}{2}$. Thus

$$S_m f(0) - \frac{1}{2}\left[f(0+) + f(0-)\right] = \underbrace{\int_0^{\frac{1}{2}} [f(x) - f(0+)] D_m(x)\,dx}_{I_m} + \underbrace{\int_{-\frac{1}{2}}^{0} [f(x) - f(0-)] D_m(x)\,dx}_{J_m}.$$

<u>Show</u> $I_m \to 0$ (Similarly $J_m \to 0$).

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(\delta) - f(0+) < \varepsilon / C$ where $C > 0$ satifies $\left|\int_a^b D_m(x)\,dx\right| < C$ $\forall [a,b] \subseteq [-\frac{1}{2}, \frac{1}{2}]$. By Lemma 8.41, $\exists \eta \in [0, \delta]$ s.t.

$$\left|\int_0^{\delta} [f(x) - f(0+)] D_m(x)\,dx\right| = [f(\delta) - f(0+)] \left|\int_\eta^{\delta} D_m(x)\,dx\right| < \varepsilon.$$

Now, $$\int_\delta^{\frac{1}{2}} [f(x) - f(0+)] D_m(x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \boxed{\chi_{[\delta, 1/2)}(x) \frac{[f(x) - f(0+)]}{\sin \pi x}} \sin(2m+1)\pi x\,dx$$

$\in L^1(\mathbb{T})$

$\longrightarrow 0$ by the Riemann-Lebesgue Lemma. So,

$$\limsup_{m \to \infty} \left|\int_0^{1/2} [f(x) - f(0+)] D_m(x)\,dx\right| < \varepsilon.$$

<u>QED</u>.

Note. If f has a jump discontinuity, then the Fourier series of f does not converge to f pointwise.

<u>The Localization Thm</u> If $f, g \in L^1(\mathbb{T})$, $f = g$ on an open interval I, then $S_m f - S_m g = S_m(f-g) \to 0$ uniformly on compact subsets of I.

<u>Corollary</u>. Let $f \in L^1(\mathbb{T})$ and I be an open interval of length ≤ 1.

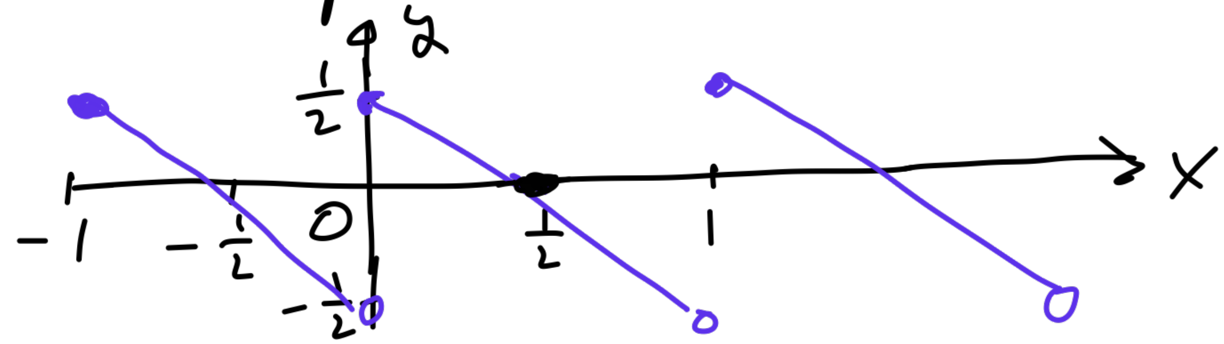
(1) If f agrees on I with some g s.t. $\hat{g} \in \ell^1(\mathbb{Z})$, then $S_m f \to f$ on compact subsets of I.

(2) If f is absolutely continuous on I and $f' \in L^p(I)$ for some $p > 1$, then $S_m f \to f$ uniformly on compact sets of I.

Ideas. If $f = g$ on I, then $S_m f - f = S_m f - g = \left(S_m f - S_m g\right) + (S_m g - g)$. $S_m g \to g$ uniformly on $\mathbb{R}$.

The Gibbs phenomenon.

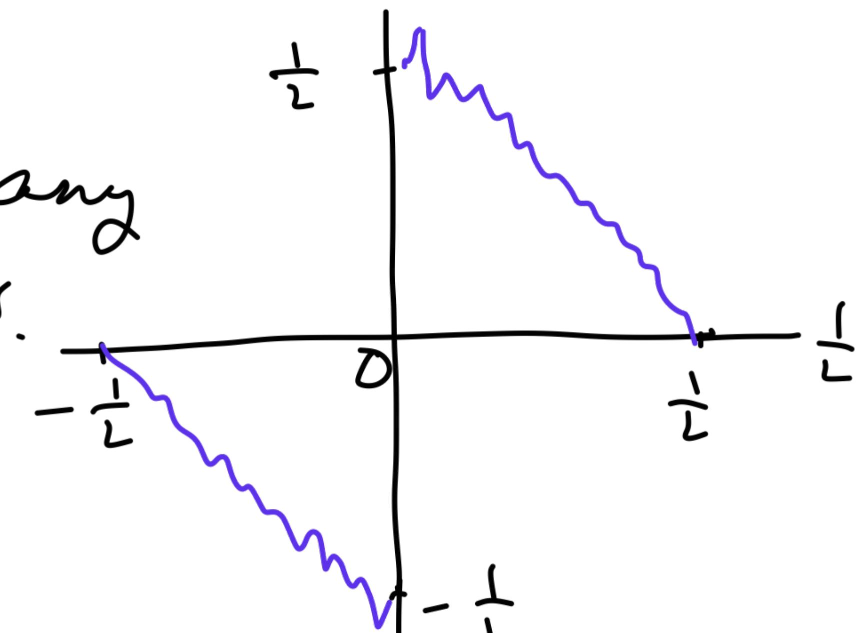
Example $\phi(x) = \frac{1}{2} - x - [x]$, $[x]$ = greatest integer $\le x$.



- $\phi \in C^\infty$ except for integer jump discont.
- $\phi(j+) - \phi(j-) = 1$.
- $\hat{\phi}(0) = 0$, $\hat{\phi}(k) = \frac{1}{2\pi i k}$ $(k \neq 0)$

$$S_m \phi(x) = \sum_{k=1}^{m} \frac{\sin 2\pi k x}{\pi k}.$$

$S_m \phi(j) = 0 \quad \forall j \in \mathbb{Z}$.



- Corollary $\Rightarrow S_m \phi \to \phi$ unif. on any compact set containg no intergers.
- Near an integer (e.g. 0.), $S_m \phi$ contains a sequence of spikes tend to 0 in width but not height.

$m \gg 1$, first max. of $S_m \phi$ on the right of 0 is ≈ 0.5895, 18% larger than $1/2 = \phi(0+)$.

Details Let $\Delta_m(x) = S_m\phi(x) - \phi(x)$. The jth crt pt of Δ_m on the right of 0 at $j/(2m+1)$. $\lim_{m\to\infty} \Delta_m\left(\frac{j}{2m+1}\right) = \frac{1}{\pi}\int_0^{j\pi} \frac{\sin t}{t} - \frac{1}{2}$.
positive if j odd. negative j: even. max. at $j=1$.