Monday, 5/18/2020 Lecture 21

Review: Ch7. Radon measures

Part I Preparations: LCH, Urysohn, partition of unity, Borel measures, Function spaces: $BC(X)$, $C_c(X)$, $C_0(X)$

⊙ X: LCH = Locally compact Hausdorff.

Urysohn's Lemma X: LCH. K (compact) $\subseteq U$ (open) $\subseteq X$

$\Longrightarrow \exists f \in C_c(X, [0,1])$ $K \prec f \prec U$.

Recall: $\varphi : X \to \mathbb{C}$. $\mathrm{supp}(\varphi) = \overline{\{x \in X : \varphi(x) \neq 0\}}$.

Partition of unity compact $K \subseteq \bigcup_{j=1}^{m} U_j$, U_j: open

$\Longrightarrow \exists g_j \prec U_j$ s.t. $\sum_{j=1}^{m} g_j = 1$ on K.

⊙ X: LCH, $\mathcal{B}_X$ = the Borel σ-alg. of X = the smallest σ-alg. containing all the open sets

A Borel measure μ is a measure on $(X, \mathcal{B}_X)$.

Def. X: LCH, μ: Borel measure on X.

- μ is inner regular at $E \in \mathcal{B}_X$, if
  $\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K : \text{compact}\}$
  μ is **inner regular**, if it is at all $E \in \mathcal{B}_X$.
- μ is outer regular at $E \in \mathcal{B}_X$, if
  $\mu(E) = \inf\{\mu(U) : U \supseteq E,\ U : \text{open}\}$
  μ is **outer regular**, if it is at all $E \in \mathcal{B}_X$.
- **regular** = inner regular + outer regular.

- **Signed measure** ν on $(X, \mathcal{B}_X)$.
  Jordan decomposition: $\nu = \nu^+ - \nu^-$. $|\nu| = \nu^+ + \nu^-$.
  Complex measure ν on $(X, \mathcal{B}_X)$: $\|\nu\| = |\nu|(X) < \infty$

  $\nu \ll |\nu|$, $\dfrac{d\nu}{d|\nu|} = 1$ ν-a.e. ($\Longleftrightarrow$ $|\nu|$-a.e.)

  $|\nu|(E) = \sup\left\{\left|\int_E f\, d\nu\right| : |f| \le 1\right\}$.

⊙ Spaces of functions $\quad X$: LCH

$B(X) = \{\text{all bounded functions } f : X \to \mathbb{C}\}$

$C(X) = \{\text{all cont. functions } f : X \to \mathbb{C}\}$

$BC(X) = B(X) \cap C(X)$

$C_0(X) = \{f \in C(X) : \text{"} f(\infty) = 0 \text{"}\}$ $\quad \boxed{\forall \varepsilon > 0. \ \{|f| \geq \varepsilon\} \text{ is compact}}$

$C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ is compact}\}$

uniform norm: $f \in B(X)$; $\|f\|_u = \sup_{x \in X} |f(x)|$

$B(X)$, $BC(X)$: Banach spaces

$$C_c(X) \subseteq \overline{C_c(X)} = C_0(X) \subseteq BC(X) \subseteq \begin{matrix} B(X) \\ C(X) \end{matrix}$$

If X is compact then

$$C_c(X) = C_0(X) = C(X).$$

Part II Radon measures / Two main theorems

- Def. X: LCH, μ: Borel meas. on X. μ is a **Radon meas.** on X if μ is
  - finite on compact sets;
  - inner regular at open sets;
  - outer regular.

- Examples
  - Lebesgue measure
  - Dirac mass on $\mathbb{R}^n$.
  - μ: Radon. $\phi \in L^1(\mu)$, $\phi \geq 0$.
    $d\nu = \phi \, d\mu \Longrightarrow \nu$ is Radon.

- μ: Radon on LCH X.
  $\operatorname{supp}(\mu)$ $=$ the complement of the union of all open sets U s.t. $\mu(U) = 0$.

The Riesz Representation Thm X: LCH. $I: C_c(X) \to \mathbb{C}$ linear + positive $\Longrightarrow$ $\exists!$ Radon meas. μ on X s.t.

$I(f) = \int_X f\,d\mu \quad \forall f \in C_c(X)$. Moreover,

$\forall U$: open $\mu(U) = \sup\{I(f) : f \in C_c(X, [0,1]),\ f \prec U\}$.

$\forall K$: compact $\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}$.

For any Radon meas. μ:

$\forall U$: open $\Rightarrow \mu(U) = \sup\{\int_X f\,d\mu : f \in C_c(X, [0,1]),\ f \prec U\}$

$\forall K$: compact $\Rightarrow \mu(K) = \inf\{\int_X f\,d\mu : f \in C_c(X),\ f \geq \chi_K\}$.

The Riesz Representation X: LCH. $M(X) \cong [C_0(X)]^*$

$\cong$: isometric isomorphis. $\mu \longmapsto I_\mu$. $I_\mu(f) = \int_X f\,d\mu \quad \forall f \in C_0(X)$

special case: X = compact Hausdorff.

$\Longrightarrow M(X) \cong [C(X)]^*$.

$\odot$ $I: C_0(X) \to \mathbb{C}$: linear + positive $\Longrightarrow$ bounded.