Part III Properties of Radon Measures

III.1 Regularity

μ: Radon meas. on LCH X.

- $E \in \mathcal{B}_X$ is μ σ-finite $\Longrightarrow$ μ is regular at E
  - σ-finite Radon $\Longrightarrow$ regular.
  - X: σ-compact $\Longrightarrow$ Radon is regular.
- The squeeze thm for a σ-finite Radon measure: $\forall E \in \mathcal{B}_X$ $\forall \varepsilon > 0$, $\exists F, U$ $F \subseteq E \subseteq U$, F: closed, U: open. $\mu(U \setminus F) < \varepsilon$.

$\forall E \in \mathcal{B}_X$: $\exists A$ (F_σ-set), B (G_δ-set). $A \subseteq E \subseteq B$. $\mu(B \setminus A) = 0$.

III.2 When a Borel is Radon?

- Thm 7.8 X: LCH. every open set is σ-compact $\Longrightarrow$ a Borel meas. μ is Radon $\Longleftrightarrow$ μ is finite on compact sets.

III.3 Approximations

- μ: Radon $\Longrightarrow$ $C_c(X)$ is dense in $L^p(\mu)$ $(1 \le p < \infty)$.
- **Lusin's Thm** X: LCH, μ: Radon. $f: X \to \mathbb{C}$ meas. $\mu(\{f \neq 0\}) < \infty$. $\Longrightarrow$ $\forall \varepsilon > 0$, $\exists \phi \in C_c(X)$ s.t. $\mu(\{\phi \neq f\}) < \varepsilon$. If $\|f\|_u < \infty$ then ϕ can be taken to satisfy $\|\phi\|_u \le \|f\|_u$.

III.4 weak-* (or vague) convergence

$\mu_n, \mu \in M(X)$: $\mu_n \xrightarrow{*} \mu$ if $\int f\, d\mu_n \to \int f\, d\mu$ $\forall f \in C_0(X)$.

- $\mu_n \xrightarrow{*} \mu \Longrightarrow \sup_{n \ge 1} \|\mu_n\| < \infty$.
- $\mu_n \xrightarrow{*} \mu \not\Longrightarrow \mu_n(E) \to \mu(E)$. $\forall E \in \mathcal{B}_X$.
- $\mu_n \xrightarrow{*} \mu \Longrightarrow$ $\forall U$ open $\liminf_{n \to \infty} \mu_n(U) \ge \mu(U)$
  $\forall K$: compact. $\limsup_{n \to \infty} \mu_n(K) \le \mu(K)$

Part IV LSC and USC Functions

Def $f : X \to (-\infty, \infty]$ is LSC if $\{f > a\}$ is open $\forall a \in \mathbb{R}$.
$f : X \to [-\infty, \infty)$ is USC if $\{f < a\}$ is open $\forall a \in \mathbb{R}$.

- $f : X \to \mathbb{C}$: f is cont. $\Rightarrow$ f is LSC $\Leftrightarrow$ $-f$ is USC
- U open $\Rightarrow$ χ_U is LSC.
- f_1, f_2 : LSC $\Rightarrow$ $f_1 + f_2$ LSC, $c \geq 0 \Rightarrow cf_1$ is LSC
- $\forall f \in \mathcal{F}$: f is LSC $\Rightarrow$ $\sup \mathcal{F}$ is LSC
- X : LCH, $f \geq 0$: LSC $\Rightarrow$ $f(x) = \sup\{g(x) : g \in C_c(X), 0 \leq g \leq f\}$

Prop X : LCH, μ : Radon

- $\mathcal{G}$: family of nonnegative LSC directed by "$\leq$"
$\Rightarrow \int \sup_{g \in \mathcal{G}} g \, d\mu = \sup_{g \in \mathcal{G}} \int g \, d\mu$
- $f \geq 0$, f : LSC $\Rightarrow$ $\int f \, d\mu = \sup\{\int g \, d\mu : g \in C_c(X), 0 \leq g \leq f\}$
- $f \geq 0$, Borel meas. $\Rightarrow$ $\int f \, d\mu = \inf\{\int g \, d\mu : g \geq f, g : LSC\}$.
$\{f > 0\}$ σ-finite $\Rightarrow$ $\int f \, d\mu = \sup\{\int g \, d\mu : 0 \leq g \leq f, g : USC\}$.

Part V Product Radon Measures

<u>Thm</u> X, Y: LCH, $\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}$

X, Y: second countable:

① $\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}$.

② μ, ν: Radon on $X, Y \Longrightarrow \mu\times\nu$ Radon on $X\times Y$.

<u>Def</u> X, Y: LCH, μ, ν: Radon meas. $\Rightarrow C_c(X\times Y) \subseteq L^1(\mu\times\nu)$.

$f \longmapsto \int f\, d(\mu\times\nu)$ is linear + positive on $C_c(X\times Y) \Rightarrow \exists!$

Radon meas. $\mu \hat{\times} \nu$, on $X\times Y$ s.t.

$$\int f\, d(\mu\hat{\times}\nu) = \int f\, d(\mu\times\nu) \quad \forall f \in C_c(X\times Y).$$

<u>Thm</u> X, Y: LCH, μ, ν: σ-finite Radon on X, Y, resp.

⊙ $E \in \mathcal{B}_{X\times Y} \Longrightarrow x \longmapsto \nu(E_x)$, $y \longmapsto \mu(E^y)$: meas.

$\mu\hat{\times}\nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$. $\quad \mu\hat{\times}\nu\big|_{\mathcal{B}_X\otimes\mathcal{B}_Y} = \mu\times\nu$.

⊙ (Fubini-Tonelli:) $f \in L^1(\mu\hat{\times}\nu)$ or $f: X\times Y \to \mathbb{C}$ meas. $f \geq 0$.

$\Longrightarrow \int f\, d(\mu\hat{\times}\nu) = \iint f\, d\mu\, d\nu = \iint f\, d\nu\, d\mu$.

Prob. Session

Prob. 4. HW#1 X: LCH, μ: Radon. $\phi \in L^1(\mu)$, $\phi \geq 0$.
$d\nu = \phi\, d\mu \Longrightarrow \nu$ is Radon.

Pf Only some part (see sol'n on class webpage)
Let $E \in \mathcal{B}_X$. Show ν is inner regular at E. (E is not necessary open.) Note: $\mu(E)$ may not be finite.

$\forall \varepsilon > 0$. $\exists \delta > 0$. $A \in \mathcal{B}_X$, $\mu(A) < \delta \Longrightarrow \nu(A) = \int_A \phi\, d\mu < \varepsilon/2$.

Let $A_k = \{\frac{1}{2k} \leq \phi \leq 2k\}$ $(k = 1, 2, \cdots)$. $A_k \in \mathcal{B}_X$. $A_k \uparrow \{\phi > 0\}$.

MCT $\Rightarrow \nu(A_k \cap E) = \int \phi \chi_{A_k \cap E}\, d\mu \to \int \phi \chi_{\{\phi > 0\} \cap E}\, d\mu$

$= \int_E \phi\, d\mu = \nu(E)$. $\Longrightarrow \exists N$. s.t. $\nu(A_N \cap E) > \nu(E) - \varepsilon/2$

Since $\mu(A_N \cap E) \leq \mu(A_N) \leq \int_{A_N} 2N \phi\, d\mu \leq \int_X 2N \phi\, d\mu < \infty$,
μ is regular at $E \Longrightarrow \exists$ compact $K \subseteq A_N \cap E \subseteq E$.
s.t. $\mu((A_N \cap E) \setminus K) < \delta$. Hence $\nu((A_N \cap E) \setminus K) < \varepsilon/2$
and $\nu(E \setminus K) \leq \nu(E \setminus (A_N \cap E)) + \nu((A_N \cap E) \setminus K) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$E \in \mathcal{B}_X \Longrightarrow E^c \in \mathcal{B}_X$. ν is inner regular at E^c

$\Longrightarrow \forall \varepsilon > 0$. $\exists$ compact $F \subseteq E^c$: $\nu(F) > \nu(E^c) - \varepsilon$.

$U = F^c$. F is closed, U is open. $U \supseteq E$. $\nu(U)$

$= \nu(F^c) = \nu(X) - \nu(F) < \nu(X) - \nu(E^c) + \varepsilon = \nu(E) + \varepsilon$.

<u>Prob. 6 HW#1</u> X: Compact + Hausdorff, μ: Radon.

$\mu(X) = 1 \Longrightarrow \exists$ compact $K \subseteq X$, s.t. $\mu(K) = 1$ and

if $H \subsetneq K$, H: compact, then $\mu(H) < 1$.

<u>Pf</u> $K = \operatorname{supp}(\mu)$: compact. $\mu(K^c) = 0$. So,

$\mu(K) + \mu(K^c) = \mu(X) = 1 \Longrightarrow \mu(K) = 1$.

Let $H \subsetneq K$, H: compact. $\exists x \in K \setminus H \Longrightarrow \exists U, V$:

open, disj. $x \in U$, $H \subseteq V$. $x \in \operatorname{Supp}(\mu) = K \Longrightarrow \mu(U) > 0$.

Since $\mu(K) = 1$, $\mu(U \cap K) = \mu(U) > 0$. Now, $H \cup (U \cap K) \subseteq K$,

$H \cap (U \cap K) = \phi \Longrightarrow \mu(H) + \mu(U \cap K) \leq \mu(K) = 1 \Longrightarrow$

$\mu(H) \leq 1 - \mu(U \cap K) < 1$.

Prob. 2, HW#2 X: LCH, μ: Radon. $\mu(\{x\}) = 0 \ \forall x \in X$.
$A \in \mathcal{B}_X$: $0 < \mu(A) < \infty$. $\Longrightarrow$ $\forall \alpha \in (0, \mu(A))$ $\exists B \in \mathcal{B}_X$, $B \subseteq A$.
$\mu(B) = \alpha$. [An intermediate-value Thm].

Pf $\exists$ compact $K \subseteq A$ s.t. $\mu(K) > \alpha$. Construct compact sets K_j $(j = 1, 2, \cdots)$ s.t.

① $K \supseteq K_1 \supseteq K_2 \supseteq \cdots$

② $\alpha \le \mu(K_j) \le \alpha + \alpha/2^j$ $(j = 1, 2, \cdots)$.

If for some j, $\alpha = \mu(K_j)$, then we choose all $K_{j+n} = K_j$ $(n = 1, 2, \cdots)$. So, we may assume $\alpha < \mu(K_j) \le \alpha + \alpha/2^j$. Once all these K_j's are constructed, then $B = \bigcap_{j=1}^{\infty} K_j \subseteq K \subseteq A$ and $\mu(B) = \lim_{j \to \infty} \mu(K_j) = \alpha$, as desired.

$\forall x \in K$. Since $\mu(\{x\})=0$ and μ is outer regular, there exists an open set $\widetilde{V}_x \ni x$ such that $\mu(\widetilde{V}_x) < \alpha/2$. Since $\{x\}$ is compact, $\{x\} \subseteq \widetilde{V}_x$, and X is locally compact and Hausdorff, there exists a precompact open set V_x such that $x \in V_x \subseteq \overline{V_x} \subseteq \widetilde{V}_x$ (cf. Prop. 4.31). Note that $\mu(\overline{V_x}) \le \mu(\widetilde{V}_x) < \alpha/2$. Now $\{V_x : x \in K\}$ covers K, which is compact. Thus, there is a finite subcover $V_{x_1}, \dots, V_{x_m}$. Let $F_j = \left(\bigcup_{i=1}^{j} \overline{V_i}\right) \cap K$, $j=1,\dots,m$. Then each F_j is compact. $F_1 \subseteq \cdots \subseteq F_m = K$, $\mu(F_1) \le \mu(\overline{V_1}) < \frac{\alpha}{2}$, $\mu(F_m) = \mu(K) > \alpha$. $\mu(F_{j+1}) - \mu(F_j) \le \mu(\overline{V_{j+1}}) < \alpha/2$ $(j=1,\dots,m-1)$. Thus, there exists j such that $\mu(F_{j-1}) \le \alpha \le \mu(F_j)$. Let $K_1 = F_j$. Then K_1 is compact, $K_1 \subseteq K$, $0 \le \mu(K_1) - \alpha \le \mu(F_j) - \mu(F_{j-1}) < \alpha/2$. Replace K by K_1, $\alpha/2$ by $\alpha/2^2$, etc, we can obtain K_2. An induction provides all K_j $(j=1,2,\dots)$ as needed.