Friday, 5/22/2020. Lecture 23

Review of Ch 8. Elements of Fourier Analysis

Part I Translations / Convolutions / Mollifiers

(or: Approximation identify)

Def $f: \mathbb{R}^n \to \mathbb{C}$. $\forall y \in \mathbb{R}^n$: $\underbrace{\tau_y}_{\text{translation}} f(x) = f(x-y)$ $\forall x \in \mathbb{R}^n$.

⊙ **Prop** $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. $\forall z \in \mathbb{R}^n \Rightarrow \lim_{y \to 0} \|\tau_{y+z} f - \tau_z f\|_p = 0$.

Def $f, g: \mathbb{R}^n \to \mathbb{C}$ measurable. $\underbrace{(f * g)}_{\text{convolution}}(x) = \int_{\mathbb{R}^n} f(x-y) g(y) dy$ if the integral exists.

See Young's inequality

⊙ **Prop**

⊙ $f * g = g * f$.

⊙ $(f*g)*h = f*(g*h)$

⊙ $\tau_z(f*g) = (\tau_z f)*g = f * \tau_z g$

⊙ $\text{supp}(f*g) \subseteq \overline{\text{supp}(f) + \text{supp}(g)}$.

⊙ $f \in L^1$, $g \in C^k$, $\partial^\alpha g$ bounded $(|\alpha| \le k) \Longrightarrow f * g \in C^k$
$\partial^\alpha (f*g) = f * \partial^\alpha g$.

⊙ Young's ineq. $f \in L^1$, $g \in L^p (1 \le p \le \infty) \Longrightarrow \|f*g\|_p \le \|f\|_1 \|g\|_p$.
Generalization: $f \in L^p$, $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \Rightarrow \|f*g\|_r \le \|f\|_p \|g\|_q$.

⊙ Prop $f \in L^p$, $g \in L^q$ p, q: conjugate $\Longrightarrow$ $f*g$: unif. cont. + bounded.
$\|f*g\|_u \le \|f\|_p \|g\|_q$. If $1 < p, q < \infty$ then $f * g \in C_0(\mathbb{R}^n)$.

⊙ Thm. $\phi \in L^1(\mathbb{R}^n)$, $\phi_t(x) = t^{-n}\phi(\frac{x}{t})$. $\int \phi \, dx = 1$. [often: $\phi \in C_c^\infty(\mathbb{R}^n)$]

⊙ $f \in L^p (1 \le p < \infty) \Longrightarrow f * \phi_t \to f$ in L^p as $t \to 0$.

⊙ $f : \mathbb{R}^n \to \mathbb{C}$ bounded and uniformly cont. $\Longrightarrow$
$f * \phi_t \to f$ unif.

⊙ $f \in L^\infty$, f is cont. on open $U \Longrightarrow f * \phi_t \to f$ unif.
on compact subsets of U

If in addition $|\phi(x)| \le C(1+|x|)^{-n-\varepsilon}$, then $f \in L^p (1 \le p \le \infty)$
$\Longrightarrow f*\phi_t(x) \longrightarrow f(x)$ $\forall x \in L_f$ (the Lebesgue set of f).

⊙ The C^∞ Urysohn Lemma $K \subseteq U \subseteq \mathbb{R}^n$, K: compact,
U: open. $\Longrightarrow \exists f \in C_c^\infty(\mathbb{R}^n, [0,1])$ s.t. $K \prec f \prec U$.

Part II. The Fourier Transform

II.1 $f \in L^1(\mathbb{T}^n)$: $\hat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x}\, dx = \langle f, E_k\rangle_{\mathbb{T}^n}$ $(k \in \mathbb{Z}^n)$

- $\|\hat{f}\|_{\ell^\infty(\mathbb{Z}^n)} \le \|f\|_{L^1(\mathbb{T}^n)}$
- $E_k(x) = e^{2\pi i k\cdot x}$. $\{E_k\}_{k\in\mathbb{Z}^n}$: orthonormal basis of $L^2(\mathbb{T}^n)$
  - $f \in L^2(\mathbb{T}^n)$: $f = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) E_k$ in $L^2(\mathbb{T}^n)$
  - $f \in L^2(\mathbb{T}^n)$: $\|f\|_{L^2(\mathbb{T}^n)} = \|\hat{f}\|_{\ell^2(\mathbb{Z}^n)} = \left(\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2\right)^{1/2}$
- The Hausdorff-Young Ineq. $1 \le p \le 2$. $f \in L^p(\mathbb{T}^n)$

  $\Longrightarrow \hat{f} \in \ell^{p'}(\mathbb{Z}^n)$, $p' = p/(p-1)$. $\|\hat{f}\|_{\ell^{p'}} \le \|f\|_{L^p}$.

II.2 $f \in L^1(\mathbb{R}^n)$: $\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i k\cdot x}\, dx$ $(\xi \in \mathbb{R}^n)$.

- Prop
  - $\hat{f} \in BC(\mathbb{R}^n)$. $\|\hat{f}\|_u \le \|f\|_1$.
  - $\widehat{f * g} = \hat{f}\hat{g}$
  - $x^\alpha f \in L^1$ $(|\alpha| \le k) \Longrightarrow \hat{f} \in C^k$. $\partial^\alpha \hat{f} = \widehat{(-2\pi i x)^\alpha f}$
  - $f \in C^k$, $\partial^\alpha f \in L^1$ $(|\alpha| \le k)$. $\partial^\alpha f \in C_0$ $(|\alpha| \le k-1) \Rightarrow \widehat{\partial^\alpha f} = (2\pi i \xi)^\alpha \hat{f}$

- The Fourier transf. of Gaussian: $g(x)=e^{-\pi x^2} \Rightarrow \hat{g}=g$.
- The Riemann-Lebesgue Lemma: $f\in L^1(\mathbb{R}^n) \Rightarrow \hat{f}\in C_0(\mathbb{R}^n)$.
- The Fourier inversion thm $f, \hat{f}\in L^1 \Rightarrow \check{\hat{f}} = \hat{\check{f}} \overset{a.e}{=} f$.

Corollary: $\mathcal{F}: L^1(\mathbb{R}^n)\to C_0(\mathbb{R}^n)$: linear + bounded, 1-1, (but not onto)

- The Plancherel Thm $f\in L^1\cap L^2 \Longrightarrow \mathcal{F}f\in L^2$. $\mathcal{F}|_{L^1\cap L^2}$ extends uniquely to a unitary isomorphism on L^2.
- The Hausdorff-Young ineq. $1\le p\le 2$, $p'=p/(p-1)$.
  $f\in L^p(\mathbb{R}^n) \Longrightarrow \hat{f}\in L^{p'}(\mathbb{R}^n)$. $\|\hat{f}\|_{p'}\le \|f\|_p$.

II.3
- Thm. $f\in L^1(\mathbb{R}^n) \Longrightarrow \sum_{k\in\mathbb{Z}^n} \tau_k f \to Pf\in L^1$ (a.e. + L^1),
  $\|Pf\|_1\le\|f\|_1$, and $\underset{\in\mathbb{T}^n}{\widehat{Pf}}(k) = \underset{\mathbb{R}^n}{\hat{f}}(k)$ $(\forall k\in\mathbb{Z}^n)$.
- The Poisson sum. $f\in C(\mathbb{R})$, $|f(\cdot)|, |\hat{f}(\cdot)|\le C(1+|\cdot|)^{-n-\varepsilon}$. $(C,\varepsilon>0)$
  $\Rightarrow \sum_k f(x+k) = \sum_k \hat{f}(k)e^{2\pi i k\cdot x}$, both conv. abs. + unif. on $\mathbb{T}^n$.

Part III The Schwartz Space $\mathcal{S}$

Def $\mathcal{S} = \{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{(N,\alpha)} < \infty \ \forall N \ \forall \alpha \} \subseteq C_0^\infty(\mathbb{R}^n)$.

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|, \quad N \geq 0,\ N \in \mathbb{Z},\ \alpha = (\alpha_1, \cdots, \alpha_n)$$

- $\mathcal{S}$ is a Fréchet space. It's topology is def. by $\{\|\cdot\|_{(N,\alpha)} : N, \alpha\}$: locally convex and Hausdorff topological vector space with a translationally invariant complete metric. p_n: seminorms:

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{p_n(f-g)}{1+p_n(f-g)} \quad \forall f, g \in \mathcal{S}.$$

- $f_n \to f$ in $\mathcal{S} \iff \|f_n - f\|_{(N,\alpha)} \to 0 \quad \forall N, \forall \alpha$.
- $f \in C^\infty$: $f \in \mathcal{S} \iff x^\beta \partial^\alpha f$ is bounded $\forall \alpha, \beta$.
  $\iff \partial^\alpha (x^\beta f)$ is bounded $\forall \alpha, \beta$.
- $f, g \in \mathcal{S} \Rightarrow f * g \in \mathcal{S}$
- $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism.

Part IV Approximations via Fourier Integrals/ Series

Defining kernels. $\Phi \in L^1 \cap C_0$, $\check{\Phi} \in L^1$, $\Phi(0) = 1$.

Def. $\phi = \check{\Phi} \in L^1$. Call ϕ a **kernel**.

⊙ $\int \phi(x)\,dx = \Phi(0) = 1$.

$\phi_t(x) = t^{-n}\phi(\frac{x}{t}) \Longrightarrow \widehat{\phi_t}(\xi) = \Phi(t\xi)$

⊙ Examples

⊙ $\Phi(\xi) = e^{-\pi|\xi|^2}$, $\phi(x) = e^{-\pi|x|^2}$ the Gauss kernel

⊙ $\Phi(\xi) = e^{-2\pi|\xi|}$. $\phi(x) = \dfrac{\Gamma(\frac{1}{2}(n+1))}{\pi^{(n+1)/2}}(1+|x|^2)^{-\frac{n+1}{2}}$ the Poisson kernel

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$. $(\mathrm{Re}\,\alpha > 0)$

$n=1$. $\phi(x) = \dfrac{1}{\pi(1+x^2)}$.

⊙ $\Phi(\xi) = \max(1-|\xi|, 0)$

$n=1$. $\phi(x) = \left(\dfrac{\sin \pi x}{\pi x}\right)^2$.

Thm $\Phi \in L^1 \cap C_0$, $\check{\Phi} \in L^1$, $\Phi(0)=1$, $\phi = \check{\Phi}$. $\underbrace{f \in L^1 + L^2}$

Def. $f^t(x) = \int \hat{f}(\xi)\, \Phi(t\xi)\, e^{2\pi i \xi \cdot x}\, dx = \widehat{f}\, \Phi(t\cdot)\,(x)$

$\Longrightarrow$ $f^t(x) = f * \phi_t(x)$. $\forall x \in \mathbb{R}^n$.

Thm $\Phi \in C(\mathbb{R}^n)$, $|\Phi(\cdot)|, |\check{\Phi}(\cdot)| \le C(1+|\cdot|)^{-n-\varepsilon}$ $(C, \varepsilon > 0)$

$\Phi(0)=1$. $\forall f \in L^1(\mathbb{T}^n)$, $t>0$, def.

$$f^t(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)\, \Phi(tk)\, e^{2\pi i k \cdot x}$$

(which conv. absolutely).

Then
- $f \in L^p \ (1 \le p < \infty) \Longrightarrow \|f^t - f\|_p \to 0$.
- $f \in C(\mathbb{T}^n) \Longrightarrow f^t \to f$ unif.
- $f \in L^1 \Longrightarrow f^t(x) \to f(x) \quad \forall x \in L_f$.

Part V Pointwise Convergence of Fourier Series

$f \in L^1(\mathbb{T})$: $f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k x}$, **Fourier series**

The mth symmetric partial sum

$$S_m f(x) = \sum_{-m}^{m} \hat{f}(k) e^{2\pi i k x} = f * D_m(x)$$

The mth Dirichlet kernel

$$D_m(x) = \sum_{-m}^{m} e^{2\pi i k x} = \frac{\sin(2m+1)\pi x}{\sin \pi x}$$

$$\|D_m\|_{L^1(\mathbb{T})} = \int_0^1 |D_m(x)|\,dx \geq \frac{2}{\pi^2} \sum_{j=1}^{2m+1} \frac{1}{j} \geq \frac{4}{\pi^2} \sum_{j=1}^{m} \frac{1}{j}.$$

$\phi_m \in C(\mathbb{T})^*$: $\phi_m(f) = S_m f(0)$ $\forall f \in C(\mathbb{T})$. $\|\phi_m\| = \|D_m\|_1$

The mth Cesàro mean of the Fourier series

$$\sigma_m f(x) = \frac{1}{m+1} \sum_{k=0}^{m} S_k f(x) = \sum_{k=-m}^{m} \frac{m+1-|k|}{m+1} \hat{f}(k) e^{2\pi i k x} = f * F_m(x)$$

The Fejér kernel

$$F_m(x) = \frac{1}{m+1} \sum_{k=0}^{m} D_k(x) = \frac{1}{m+1} \left[\frac{\sin(m+1)\pi x}{\sin \pi x} \right]^2.$$

Thm (Pointwise convergence of Fourier Series)

$$f \in BV(\mathbb{T}) \Longrightarrow \lim_{m\to\infty} S_m f(x) = \frac{1}{2}[f(x-)+f(x+)]$$

In particular, f is cont. at $x \Longrightarrow f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$.

The localization thm $f, g \in L^1(\mathbb{T})$, $f = g$ on an open interval $I \Longrightarrow S_m(f-g) \to 0$ unif. on any compact subsets of I.

Corollary $f \in L^1(\mathbb{T})$, I: open interval of length ≤ 1.

- $f = g$ on I, $\hat{g} \in \ell^1(\mathbb{Z}) \Longrightarrow S_m f \to f$ unif. on compact subsets of I.
- f is absolutely cont. $f' \in L^p(I)$, $p > 1 \Longrightarrow S_m f \to f$ unif. on compact subsets of I.

The Gibbs phenomenon

If f has a jump discontinuity at x_0, then $S_m f(x)$ over/under estimate $S_m f(x_0)$ ~~for~~ x close to x_0, $m \gg 1$.