Wednesday, 4/1/2020, Lecture 2

The Riesz Representation Thm

- X: LCH.
- $I: C_c(X) \to \mathbb{C}$ linear and positive.

$\Longrightarrow$ $\exists!$ Radon measure μ on X s.t.

$$I(f) = \int_X f\,d\mu \qquad \forall f \in C_c(X).$$

Moreover,

(*) $\forall U$: open. $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$,

(**) $\forall K$: compact. $\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}$.

- R-measures = B-measures: $\begin{cases} \text{- finite on compact} \\ \text{- inner reg. on open} \\ \text{- outer reg.} \end{cases}$
- A method of constructing measures. But still need help from Carathéodory!

Pf of Thm Uniqueness

Claim: μ: Radon, $I(f)=\int_X f\,d\mu$ $\forall f\in C_c(X) \implies (*)$ is true:

$(*)$ $\forall U$: open. $\mu(U) = \sup\{I(f) : f\in C_c(X),\ f \prec U\}$

The claim $\implies$ μ is uniquely determined by I, since μ is outer regular at any $E\in\mathcal{B}_X$:

$$\mu(E) = \inf\{\mu(U) : U \text{ open},\ U\supseteq E\}.$$

Pf of $(*)$ $\forall f \prec U \implies I(f) = \int_X f\,d\mu = \int_U f\,d\mu \le \int_U d\mu = \mu(U)$.

$\implies \mu(U) \ge \sup\{\dots\}$

μ is Radon $\implies$ μ is inner reg. at open U, i.e.,

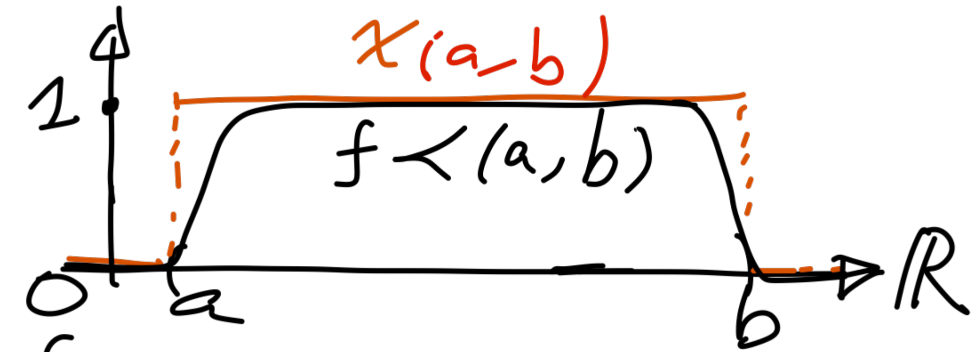
$$\mu(U) = \sup\{\mu(K) : K\subseteq U,\ K \text{ compact}\}.$$

$\forall K$: compact, $K\subseteq U \xrightarrow{\text{Urysohn}} \exists\ K \prec f \prec U$.

$\implies I(f) = \int_X f\,d\mu \ge \int_K f\,d\mu = \mu(K)$

$\implies \mu(U) = \sup\{\mu(K) : K\subseteq U,\ K\text{: compact}\} \le \sup\{\dots\}$.

Existence.



Define

(*) $\forall U$: open. $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$.

$\forall E \subseteq X$: $\mu^*(E) = \inf\{\mu(U) : U \supseteq E,\ U\text{: open}\}$.

Facts: • $U \subseteq V$ open $\Longrightarrow \mu(U) \le \mu(V)$.

• $\mu^*(U) = \mu(U)$ if U is open.

Outline

$\boxed{\text{Step 1}}$ show: μ^* is an outer measure.

Carathéodory: • $\mathcal{M}^* \triangleq \{\mu^*\text{-meas. subsets of } X\}$: σ-alg.

• $\mu^*|_{\mathcal{M}^*}$ is a measure on $\mathcal{M}^*$.

$\boxed{\text{step 2}}$ show: $\mathcal{B}_X \subseteq \mathcal{M}^*$, i.e., U: open $\Longrightarrow U \in \mathcal{M}^*$.

This implies: ① $\mu \triangleq \mu^*|_{\mathcal{B}_X}$ is a Borel measure.

② μ is outer reg. + μ satisfies (*).

$\boxed{\text{Step 3}}$ Show $\qquad \{\cdots\} \neq \phi$ by Urysohn's $\qquad \boxed{11}$

$$(**)\ \forall K\text{: comp.}: \mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}.$$

This implies:

(3) μ is finite on compact subsets. (clear!)

(4) μ is inner reg. at an open U, i.e.,

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K\text{: compact}\}.$$

Now, (1) – (4) $\Longrightarrow \mu$ is Radon, $(*)$, $(**)$ are true.

pf of (4) $\forall \alpha < \mu(U) \overset{(*)}{=} \sup\{I(f) : f \in C_c(X),\ f \prec U\}$

$\exists f \prec U$ s.t. $I(f) > \alpha$. Let $K = \operatorname{supp}(f) \subseteq U$, K: compact.

Now, $\mu(K)$ is given by $(**)$. If $g \in C_c(X)$, $g \geq \chi_K$, then

$g \geq f$ on $X \Longrightarrow I(g) \geq I(f) > \alpha \Longrightarrow \mu(K) \overset{(**)}{\geq} I(g) > \alpha$.

Hence $\mu(U) \leq \sup\{\ldots\}$. Obviously $\mu(U) \geq \sup\{\ldots\}$

$\boxed{\text{Step 4}}$ $I(f) = \int_X f\, d\mu \quad \forall f \in C_c(X)$.