Friday, 4/3/2020, Lecture 3

The Riesz Representation Thm If X is an LCH space and $I: C_c(X) \to \mathbb{C}$ is linear and positive, then $\exists!$ Radon measure μ on X s.t.

$$I(f) = \int_X f \, d\mu \qquad \forall f \in C_c(X).$$

Moreover,

(*) $\forall U$: open. $\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\}$,

(**) $\forall K$: compact. $\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}$.

Proof of Existence of such μ

- $\forall U$: open, define $\mu(U)$ by (*).
- $\forall E \subseteq X$, define

$$\mu^*(E) = \inf\{\mu(U) : U \text{ open}, U \supseteq E\}.$$

Note: U open $\Longrightarrow \mu^*(U) = \mu(U)$.

Outline

Step 1 Show: μ^* is an outer measure.

Step 2 Show: $\mathcal{B}_X \subseteq \mathcal{M}^*$ = σ-alg. of μ^*-meas sets

i.e., show U: open $\Longrightarrow$ U is μ^*-measurable.

This implies:
① $\mu \triangleq \mu^*|_{\mathcal{B}_X}$ is a Borel measure.
② μ is outer reg. + μ satisfies $(*)$.

Step 3 Show

$(**)$ $\forall K$: comp.: $\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}$.

This implies:
③ μ is finite on compact subsets.
④ μ is inner reg. at an open U.

Now, ① – ④ $\Longrightarrow$ μ is Radon, $(*)$, $(**)$ are true.

Step 4 Show $I(f) = \int_X f\,d\mu$ $\forall f \in C_c(X)$.

Recall
⊙ (*) U open: $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$
⊙ $E \subseteq X$: $\mu^*(E) = \inf\{\mu(U) : U \supseteq E,\ U\text{: open}\}$

Step 1: show: μ^* is an outer measure.

Step 1.1 Claim: U_j open $\Longrightarrow \mu(\bigcup_{j=1}^{\infty} U_j) \leq \sum_{j=1}^{\infty} \mu(U_j)$.

Pf. Let $U = \bigcup_{j=1}^{\infty} U_j$ open. $\forall f \prec U$. Let $K = \operatorname{supp}(f) \subseteq U$, K: compact $\Rightarrow K \subseteq \bigcup_{j=1}^{n} U_j$. Partition of unity: $\exists g_j \prec U_j$, $\sum_{j=1}^{n} g_j = 1$ on $K \Longrightarrow f = \sum_{j=1}^{n} f g_j$ on X and $f g_j \prec U_j$. Thus

$$I(f) = \sum_{j=1}^{n} I(f g_j) \overset{(*)}{\leq} \sum_{j=1}^{n} \mu(U_j) \Longrightarrow \mu(U) \overset{(*)}{\leq} \sum_{j=1}^{\infty} \mu(U_j).$$

Step 1.2 Def. $\mu^{**}(E) = \inf\{\sum_{j=1}^{\infty} \mu(U_j) : U_j \text{ open},\ E \subseteq \bigcup_{j=1}^{\infty} U_j\}$.

μ^{**} is an outer measure (Prop. 1.10).

$\forall U_j$ open, $E \subseteq \bigcup_{j=1}^{\infty} U_j \Longrightarrow \mu^*(E) \underset{\text{def. of } \mu^*(E)}{\leq} \mu(\bigcup_{j=1}^{\infty} U_j) \overset{\text{step 1.1}}{\leq} \sum_{j=1}^{\infty} \mu(U_j)$

Hence, $\mu^*(E) \leq \mu^{**}(E)$

$\forall U$ open, $U \supseteq E$. Let $U_1 = U$, $U_j = \phi$ ($j \geq 2$). Then,

$\mu(U) = \sum_{j=1}^{\infty} \mu(U_j) \geq \mu^{**}(E) \Longrightarrow \mu^*(E) \geq \mu^{**}(E)$.

Recall

(*) U open: $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$

$E \subseteq X$: $\mu^*(E) = \inf\{\mu(U) : U \supseteq E,\ U:\text{open}\}$

Step 2 Show: U open $\Rightarrow$ U is μ^*-measurable.

i.e., $E \subseteq X$, $\mu^*(E) < \infty \Longrightarrow \mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$.

Step 2.1 E is open. So, $E \cap U$ is open. $\forall \varepsilon > 0$:

Def. of $\mu(E \cap U) \Longrightarrow \exists f \prec E \cap U$ s.t. $I(f) > \mu(E \cap U) - \varepsilon$.

Also, $E \setminus \operatorname{supp}(f)$ is open $\overset{(*)}{\Longrightarrow} \exists g \prec E \setminus \operatorname{supp}(f)$ s.t.

$I(g) > \mu(E \setminus \operatorname{supp}(f)) - \varepsilon$. Now, $f + g \prec E$, since

$\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$. Thus,

$$\mu^*(E) \underset{E:\text{open}}{=} \mu(E) \geq I(f+g) = I(f) + I(g)$$

$$> \mu(E \cap U) + \mu(E \setminus \operatorname{supp}(f)) - 2\varepsilon$$

$$\geq \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Step 2.2 General $E \subseteq X$. Def. of $\mu^*(E) \Longrightarrow \exists V$: open,

$V \supseteq E$, and $\mu(V) < \mu^*(E) + \varepsilon$. Hence,

$$\mu^*(E) + \varepsilon > \mu(V) \underset{\text{step 2.1}}{\geq} \mu^*(V \cap U) + \mu^*(V \setminus U) \geq \mu^*(E \cap U) + \mu^*(E \setminus U).$$