Monday, 4/6/2020, Lecture 4

<u>The Riesz Representation Thm</u> If X is an LCH space and $I : C_c(X) \to \mathbb{C}$ is linear and positive, then $\exists !$ Radon measure μ on X s.t.

$$I(f) = \int_X f \, d\mu \qquad \forall f \in C_c(X).$$

Moreover,

(*) $\forall U$: open. $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$,

(**) $\forall K$: compact. $\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}$.

Proof of Existence of such μ

- $\forall U$: open, define $\mu(U)$ by (*).
- $\forall E \subseteq X$, define
$$\mu^*(E) = \inf\{\mu(U) : U \text{ open},\ U \supseteq E\}.$$

Note: U open $\Longrightarrow \mu^*(U) = \mu(U)$.

Outline (Done with Steps 1 and 2.)

Step 1 Show: μ^* is an outer measure.

Step 2 Show: $\mathcal{B}_X \subseteq \mathcal{M}^* =$ σ-alg. of μ^*-meas sets

i.e., show U: open $\Longrightarrow$ U is μ^*-measurable.

This implies:
1. $\mu \triangleq \mu^*|_{\mathcal{B}_X}$ is a Borel measure.
2. μ is outer reg. + μ satisfies $(*)$.

Step 3 Show

$(**)$ $\forall K$: comp.: $\mu(K) = \inf\{I(f) : f \in C_c(X),\ f \geq \chi_K\}$.

This implies:

3. μ is finite on compact subsets.
4. μ is inner reg. at an open U.

Now, ① – ④ $\Longrightarrow$ μ is Radon, $(*)$, $(**)$ are true.

Step 4 Show $I(f) = \int_X f\,d\mu$ $\forall f \in C_c(X)$.

Recall: (*) U open: $\mu(U) = \sup\{I(f) : f \in C_c(X),\ f \prec U\}$

<u>Step 3</u> Show: (**) $\forall K$: compact. $\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}$.

Let $f \in C_c(X)$, $f \geq \chi_K$. $\forall\, 0 < \varepsilon < 1$. Let $U_\varepsilon = \{f > 1-\varepsilon\} \subseteq X$.

U_ε is open and $K \subseteq U_\varepsilon$ (since $f \geq \chi_K$).

$$\forall g \prec U_\varepsilon \Rightarrow (1-\varepsilon)^{-1} f - g \geq 0 \Longrightarrow I(g) \leq (1-\varepsilon)^{-1} I(f)$$

I is linear & positive

$\overset{(*)}{\Longrightarrow} \mu(U_\varepsilon) \leq (1-\varepsilon)^{-1} I(f)$.

$\Longrightarrow \mu(K) \leq \mu(U_\varepsilon) \leq (1-\varepsilon)^{-1} I(f) \underset{\varepsilon \to 0}{\Longrightarrow} \mu(K) \leq I(f)$

$\Longrightarrow \mu(K) \leq \inf\{\ldots\}$

Now, μ is outer regular (at K), i.e.,

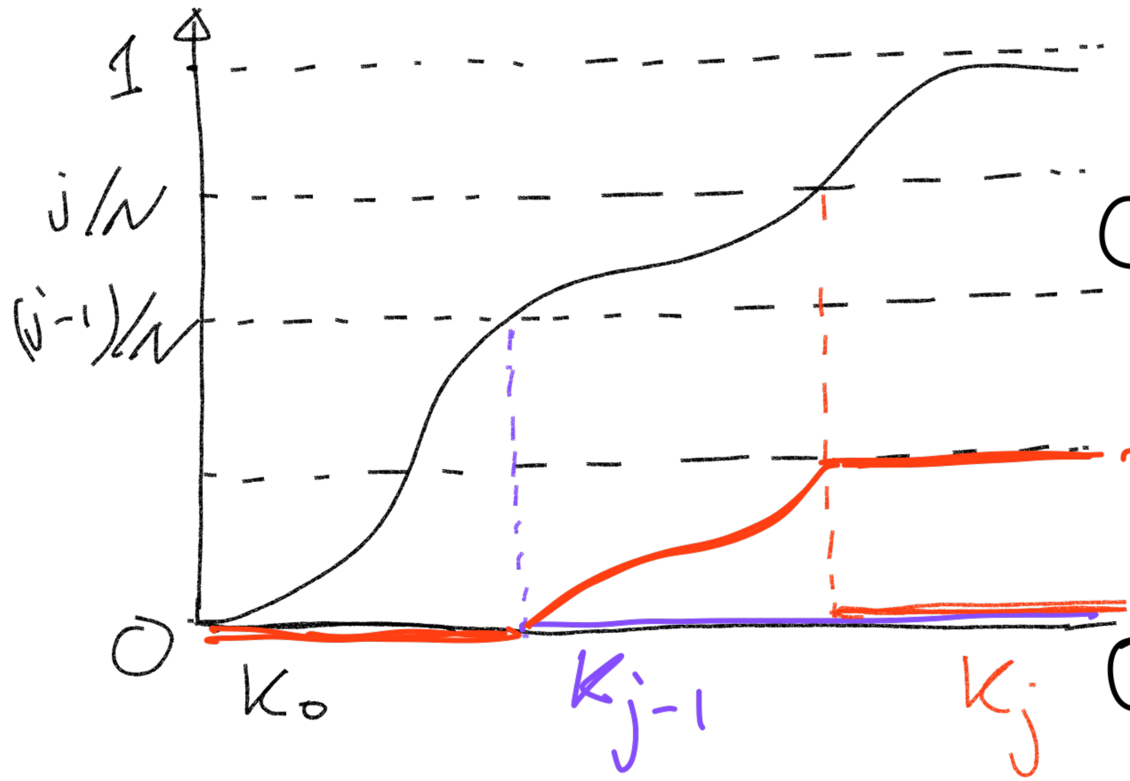
(***) $\mu(K) = \inf\{\mu(U) : U \text{ open}, U \supseteq K\}$.

$\forall U$: open, $U \supseteq K$. Urysohn's $\Longrightarrow \exists f \in C_c(X)$: $K \prec f \prec U$.

$\overset{(*)}{\Longrightarrow} I(f) \leq \mu(U) \overset{(***)}{\Longrightarrow} \mu(K) \geq I(f) \Longrightarrow \mu(K) \geq \inf\{\ldots\}$

Step 4 Show $I(f) = \int_X f\,d\mu \quad \forall f \in C_c(X)$.

Suffice to consider $f \in C_c(X, [0,1])$ which spans $C_c(X)$.



Let $N \in \mathbb{N}$. Define

$K_0 = \mathrm{supp}(f)$, $K_j = \{x \in X : f(x) \geq \frac{j}{N}\}$ $(1 \leq j \leq N)$

⊙ $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_N$, K_j : compact.

Define

$$f_j(x) = \begin{cases} 0 & \text{if } x \notin K_{j-1} \\ f(x) - \frac{j-1}{N} & \text{if } x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & \text{if } x \in K_j \end{cases}$$

⊙ $\frac{1}{N}\chi_{K_j} \leq f_j \leq \frac{1}{N}\chi_{K_{j-1}}$ $(j = 1, \cdots, N)$

⊙ $f = \sum_{j=1}^{N} f_j$ on X

$$\Longrightarrow \frac{1}{N}\mu(K_j) \leq \int f_j\,d\mu \leq \frac{1}{N}\mu(K_{j-1})$$

$$\frac{1}{N}\mu(K_j) \underset{(**)}{\leq} I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

$\forall U$: open, $U \supseteq K_{j-1}$

$\Longrightarrow N f_j \prec U$

$\Longrightarrow I(f_j) \leq \frac{1}{N}\mu(U)$

outer reg. at K_{j-1}

$$\Longrightarrow \frac{1}{N}\sum_{1}^{N}\mu(K_j) \leq \begin{matrix}\int f\,d\mu \\ I(f)\end{matrix} \leq \frac{1}{N}\sum_{0}^{N-1}\mu(K_j)$$

$$\Longrightarrow \left| I(f) - \int f\,d\mu \right| \leq \frac{\mu(K_0) - \mu(K_N)}{N} \leq \frac{\mu(\mathrm{supp}(f))}{N} \longrightarrow 0 \quad \text{Q.E.D.}$$