§7.2 Regularity and Approximation Theorems

Prop 7.5 X: LCH, μ: Radon measure on X, $E \in \mathcal{B}_X$.
μ is σ-finite on E (i.e., $E = \bigcup_{j\ge1}^{\infty} E_j$, $E_j \in \mathcal{B}_X$, $\mu(E_j) < \infty$).
$\Longrightarrow$ μ is inner regular at E, i.e.,

$$\mu(E) = \sup\{\mu(K) : K \text{ compact}, K \subseteq E\}.$$

Pf (1) Assume $\mu(E) < \infty$. Clearly, $\mu(E) \geq \sup\{\cdots\}$.
Let $\varepsilon > 0$. Since μ is outer regular at E,

$$\mu(E) = \inf\{\mu(U) : U \text{ open}, U \supseteq E\},$$

$\exists$ open $U \supseteq E$ s.t. $\mu(U) < \mu(E) + \varepsilon$. i.e., $\mu(U \setminus E) < \varepsilon$.
Since μ is inner regular at U, $\exists$ compact $F \subseteq U$,
s.t. $\mu(F) > \mu(U) - \varepsilon$. Since μ is outer reg.
at $U \setminus E$, $\exists$ open $V \supseteq U \setminus E$ s.t. $\mu(V) < \varepsilon$
Def. $K = F \setminus V$: compact. $x \in K \Rightarrow x \in F \subseteq U$, $x \notin V$
$\supseteq U \setminus E \Longrightarrow x \in U \supseteq E$, $x \notin U \setminus E \Longrightarrow x \in E \Longrightarrow K \subseteq E$.

$$\mu(K) = \mu(F \setminus V) = \mu(F) - \mu(F \setminus V)$$
$$\geq \mu(U) - \varepsilon - \mu(V) = \mu(U) - 2\varepsilon.$$

Hence, $\mu(K) = \sup\{\mu(U) : U \text{ open}, U \supseteq K\}$.

(2) Assume $\mu(E) = \infty$. Then $E = \bigcup_{j=1}^{\infty} E_j$, $E_j \in \mathcal{B}_X$, $E_j \uparrow$, $\mu(E_j) < \infty$, $\mu(E_j) \to \mu(E) = \infty$. $\forall N \in \mathbb{N}$, $\exists j \in \mathbb{N}$ s.t. $\mu(E_j) > N$. By (1), $\exists$ compact $K \subseteq E_j \subseteq E$, s.t. $\mu(K) > N$. Thus, $\sup\{\mu(K) : E \supseteq K \text{ compact}\} = \infty$.

Corollary. (1) σ-finite Radon measure $\Rightarrow$ regular.

(2) X: σ-compact (i.e., $X = \bigcup_{j=1}^{\infty} K_j$, K_j: compact).
μ: Radon $\Longrightarrow$ μ is σ-finite $\Longrightarrow$ μ is regular.

Prop 7.7 X: LCH, μ: σ-finite Radon meas. on X. $E \in \mathcal{B}_X$.

(1) $\forall \varepsilon > 0$, $\exists$ open U, close F, s.t. $F \subseteq E \subseteq U$, $\mu(U \setminus F) < \varepsilon$.

(2) $\exists$ F_σ set A, G_δ set B s.t. $A \subseteq E \subseteq B$, $\mu(B \setminus A) = 0$.

The "squeeze" Thm for measures. Pf Exercise.

Thm X: LCH. Open sets are σ-compact. μ: Borel measure on X. μ is finite on compact sets
$\Longrightarrow$ μ is regular, hence Radon. (+ regular).

$\odot$ X: LCH, 2nd countable (i.e., X has a countable base)
$\Longrightarrow$ open sets are σ-compact. (Prob.#, HW#8, Math 240B)

Pf of Thm Note $C_c(X) \subseteq L^1(\mu)$ [Why?]. Def $I: C_c(X) \to \mathbb{C}$:
$I(f) = \int_X f \, d\mu \quad \forall f \in C_c(X)$. I: linear and positive.
$\Longrightarrow \exists!$ Radon meas. ν on X, s.t. $I(f) = \int_X f \, d\nu \quad \forall f \in C_c(X)$.
$\forall U$: open, $U = \bigcup_{j=1}^{\infty} K_j$, K_j: compact, $K_j \uparrow$. Urysohn's:

$$\left.\begin{aligned} &\exists\, K_1 \prec f_1 \prec U, \\ &\exists\, K_2 \cup \operatorname{supp}(f_1) \prec f_2 \prec U. \\ &\cdots \\ &\exists\, K_n \cup \operatorname{supp}(f_{n-1}) \prec f_n \prec U. \end{aligned}\right\} \Longrightarrow \begin{aligned} &f_n \uparrow \chi_U \Rightarrow \mu(U) = \int \chi_U \, d\mu \\ &= \int \lim_n f_n \, d\mu \overset{MCT}{=} \lim_n \int f_n \, d\mu \\ &= \lim_n \int f_n \, d\nu \overset{MCT}{=} \int \lim_n f_n \, d\nu \\ &= \int \chi_U \, d\nu = \nu(U). \end{aligned}$$

$$\boxed{U: \text{open} \Rightarrow \mu(U) = \nu(U)}$$

$\forall E \in \mathcal{B}_X$ and $\forall \varepsilon > 0$. X = σ-compact $\xRightarrow{\text{Coroll.}}$ the Radon meas. ν is σ-finite $\xRightarrow{\text{Prop 7.7}}$ $\exists$ open V, closed F s.t. $F \subseteq E \subseteq V$ and $\nu(V \setminus F) < \varepsilon$. $V \setminus F$ is open. So, $\mu(V \setminus F) = \nu(V \setminus F) < \varepsilon$. i.e., $\mu(V) < \mu(F) + \varepsilon \leq \mu(E) + \varepsilon$.
$\Longrightarrow$ μ is outer reg. at E. Also, $\mu(F) > \mu(V) - \varepsilon \geq \mu(E) - \varepsilon$. F is σ-compact as X is. $\Longrightarrow$ $\exists$ compact $K_j \uparrow F$. $\mu(K_j) \to \mu(F) > \mu(E) - \varepsilon$. $\Longrightarrow$ μ is inner reg. at E $\Longrightarrow$ μ is regular. $\Longrightarrow$ μ is Radon. $\Longrightarrow \mu = \nu$.

<u>Prop 7.9</u> X: LCH, μ: Radon, $1 \leq p < \infty \Longrightarrow C_c(X)$ is dense in $L^p(\mu)$.

<u>pf</u> Simple functions $\sum_1^n a_j \chi_{E_j}$ $(\mu(E_j) < \infty)$ are dense in $L^p(\mu)$. $\forall E \in \mathcal{B}_X$, $\mu(E) < \infty$. <u>Prop. 7.5</u> $\Rightarrow$ μ is reg. at E. $\forall \varepsilon > 0$. $\exists K$: compact, U: open. s.t. $K \subseteq E \subseteq U$, $\mu(U \setminus K) < \varepsilon$.
Urysohn's $\Rightarrow \exists K \prec f \prec U \Longrightarrow \chi_K \leq f \leq \chi_U \Longrightarrow \|\chi_E - f\|_p^p \leq \mu(U \setminus K) < \varepsilon$.

Lusin's Thm X: LCH, μ: Radon, $f: X \to \mathbb{C}$ measurable.
$\mu(\{f \neq 0\}) < \infty \Longrightarrow \forall \varepsilon > 0,\ \exists \phi \in C_c(X)$ s.t. $\mu(\{f \neq \phi\}) < \varepsilon$.
If f is bounded then ϕ can satisfy $\|\phi\|_u \le \|f\|_u$.

Pf. Let $E = \{f \neq 0\}$. $\mu(E) < \infty$. Assume f is bounded.
Then $f \in L^1(\mu) \Longrightarrow \exists g_n \in C_c(X)$ s.t. $g_n \xrightarrow{L^1(\mu)} f \Longrightarrow \exists g_{n_k} \xrightarrow{a.e.} f$.
Egoroff's Thm $\Longrightarrow \exists A \in \mathcal{B}_X$, $A \subseteq E$, s.t. $\mu(E \setminus A) < \varepsilon/3$ and
$g_{n_k} \to f$ unif. on A.
$\mu(A) \le \mu(E) < \infty \Longrightarrow \mu$ is reg. at $A \Longrightarrow \exists$ compact B,
open U s.t. $B \subseteq A \subseteq U$, $\mu(U \setminus B) < \varepsilon/3$, $\mu(U \setminus E) < \varepsilon/3$.
$g_{n_k} \to f$ unif. on $A \supseteq B \Longrightarrow f|_B$ is cont. $\xRightarrow[4.34]{\text{Tietze}} \exists h \in C_c(X)$
s.t. $h = f$ on B, $\operatorname{supp}(h) \subseteq U$. Now, $\{f \neq h\} \subseteq U \setminus B$, $\mu(\{f \neq h\}) < \varepsilon$.
Def. $\beta: \mathbb{C} \to \mathbb{C}$: $\beta(z) = z$ if $|z| \le \|f\|_u$, $\beta(z) = \|f\|_u \operatorname{sgn} z$
if $|z| > \|f\|_u$. Set $\phi = \beta \circ h \in C_c(X)$. [since β is cont., $\beta(0) = 0$.]
$\|\phi\|_u \le \|f\|_u$, and $\phi = f$ on $\{h = f\}$.
If f is unbounded, then $A_n = \{0 < |f| \le n\} \uparrow E$ $\exists n \gg 1$
s.t. $\mu(E \setminus A_n) < \varepsilon/2$. So, $\exists \phi \in C_c(X)$, s.t. $\mu(\{\phi \neq \chi_{A_n} f\}) < \varepsilon/2$.
and $\mu(\{\phi \neq f\}) < \varepsilon$. Q.E.D.