Friday, 4/10/2020, Lecture 6

Topic: Semicontinuous functions. (§7.2)

Def. X: a topological space.

- $f: X \to (-\infty, \infty]$ is lower semicontinuous (LSC), if $\{f > \alpha\}$ is open $\forall \alpha \in \mathbb{R}$.
- $f: X \to [-\infty, \infty)$ is upper semicontinuous (USC), if $\{f < \alpha\}$ is open $\forall \alpha \in \mathbb{R}$.

Remarks

- LSC or USC $\Longrightarrow$ Borel measurable.
- LSC + USC $\Longleftrightarrow$ continuity.
- X: metric space:

$$f: \text{LSC} \Longleftrightarrow \liminf_{x \to x_0} f(x) \geq f(x_0)$$

$$f: \text{USC} \Longleftrightarrow \limsup_{x \to x_0} f(x) \leq f(x_0).$$

X: General topological space, use nets.

Prop 7.11 ① U open $\Longrightarrow \chi_U$ is LSC.
K compact $\Longrightarrow -\chi_K$ is LSC; χ_K is USC.

② f is LSC, $c \geq 0 \Longrightarrow cf$ is LSC.

③ f_1, f_2 are LSC $\Longrightarrow f_1 + f_2$ is LSC.

Pf $\forall \alpha > 0$. If $x_0 \in X$ and $f_1(x_0) + f_2(x_0) > \alpha$ then choose $\varepsilon > 0$ s.t. $f_1(x_0) > \alpha - f_2(x_0) + \varepsilon$. Thus,

$$\{x : f_1(x) + f_2(x) > \alpha\} \supseteq \{x : f_1(x) > \alpha - f_2(x_0) + \varepsilon\} \cap \{x : f_2(x) > f_2(x_0) - \varepsilon\}.$$

The r.h.s. is a nbh of x_0. Hence $\{f_1 + f_2 > \alpha\}$ is open.

④ $\mathcal{G}$ is a family of LSC functions. $f(x) = \sup\{g(x) : g \in \mathcal{G}\}$
$\Longrightarrow f$ is LSC. [Since: $f^{-1}((a,\infty]) = \bigcup_{g \in \mathcal{G}} g^{-1}((a,\infty])$.]

⑤ X: LCH, $f \geq 0$ LSC. Then

$$\forall x \in X: f(x) = \sup\{g(x) : g \in C_c(X),\ 0 \leq g \leq f\}$$

Pf Let $f(x) > 0$ and $0 < \alpha < f(x)$. Then, $U = \{f > \alpha\}$ is open and $x \in U$. $K := \{x\}$ (compact) $\subseteq U$ (open).
Urysohn's $\Longrightarrow \exists f \in C_c(X)$ s.t. $f(x) = \alpha$ and $0 \leq g \leq \alpha \chi_U \leq f$.
If $f(x) = 0$ the result is trivial. QED

Prop 7.12 X: LCH, $\mathcal{G}$: a family of nonnegative LSC functions directed by $\le$ (i.e., $\forall g_1, g_2 \in \mathcal{G}\ \exists g \in \mathcal{G}$ s.t. $g \le g_1$ and $g \le g_2$). Let $f = \sup\{g : g \in \mathcal{G}\}$. If μ is a Radon measure on X, then $\int f\,d\mu = \sup\{\int g\,d\mu : g \in \mathcal{G}\}$.

☉ $\int \sup = \sup \int$.

Pf By Prop. 7.11, f is LSC, hence Borel meas.

Clearly $\int f\,d\mu \ge \sup\{\int g\,d\mu : g \in \mathcal{G}\}$. show reverse

(1) $\forall \alpha < \int f\,d\mu$. Let $\phi_n = 2^{-n}\sum_{j=1}^{2^{2n}} \chi_{U_{nj}}$, $U_{nj} = \{f > j2^{-n}\}$: open. ϕ_n: simple, $0 \le \phi_n \uparrow f$, $\phi_n < f$. By MCT,

$$\lim_{n\to\infty}\int \phi_n\,d\mu = \int \lim_n \phi_n\,d\mu = \int f\,d\mu > \alpha.$$

$\Longrightarrow \exists n$. s.t. $\alpha < \int \phi_n\,d\mu = 2^{-n}\sum_{j=1}^{2^{2n}} \chi_{U_{nj}}$

(2) μ: Radon $\Longrightarrow$ μ: inner reg at open U_{nj}.

$\Longrightarrow \exists$ compact $K_j \subseteq$ (open) U_j s.t.

$$2^{-n}\sum_j \mu(K_j) > \alpha.$$

Def $\psi = 2^{-n}\sum_j \chi_{K_j} \Rightarrow \psi \le \phi_n < f$, $\int \psi\,d\mu = 2^{-n}\sum_j \mu(K_j) > \alpha$.

Show: $\exists g \in \mathcal{G}$ s.t. $g \ge \psi$. Then $\int g\,d\mu \ge \int \psi\,d\mu > \alpha$.

(3) $\forall x \in \bigcup_j K_j$, $f(x) > \phi_n(x) \ge \psi(x)$. $\exists g_x \in \mathcal{G}$ s.t. $g_x(x) > \psi(x)$.

But $-\chi_{K_j}$ is LSC. $\Rightarrow -\psi$ is LSC. $\Rightarrow g_x - \psi$ is LSC.

$\Rightarrow V_x = \{y : \psi(y) < g_x(y)\}$ is open $\Rightarrow \{V_x : x \in \bigcup_j K_j\}$ is an open cover of $\bigcup_j K_j$ (compact) $\Rightarrow \exists V_{x_1}, \dots, V_{x_m}$ s.t. $\bigcup_{k=1}^m V_{x_k} \supseteq \bigcup_j K_j$. Now, $\exists g \in \mathcal{G}$ s.t. $g_{x_k} \le g$ $(k=1,\dots,m)$. Then $\psi \le g \Rightarrow \alpha < \int \psi\,d\mu \le \int g\,d\mu$. QED

Corollary X: LCH, μ: Radon measure on X. $f \ge 0$, LSC on X. Then

$$\int f\,d\mu = \sup\left\{\int g\,d\mu : g \in C_c(X),\ 0 \le g \le f\right\}.$$

Prop 7.14 X: LCH, μ: Radon meas. on X. $f \geq 0$: Borel meas. on X. Then

$$\int f\,d\mu = \inf\left\{\int g\,d\mu : g \geq f \text{ and } g \text{ is LSC}\right\}. \quad \cdots\cdots (1)$$

If $\{f>0\}$ is σ-finite then

$$\int f\,d\mu = \sup\left\{\int g\,d\mu : 0 \leq g \leq f \text{ and } g \text{ is USC}\right\}. \cdots (2)$$

Pf $\exists\, \phi_n \geq 0$: simple, $\phi_n \uparrow f$. $\Longrightarrow f = \phi_1 + \sum_2^\infty (\underbrace{\phi_n - \phi_{n-1}}_{\geq 0}) \Longrightarrow$

$f = \sum_1^\infty a_j \chi_{E_j}$ $(a_j > 0)$. $\forall \varepsilon > 0$. $\forall j$. $\exists$ open $U_j \supseteq E_j$ s.t.

$\mu(U_j) \leq \mu(E_j) + \varepsilon/(2^j a_j)$. $\Longrightarrow g = \sum_1^\infty a_j \chi_{U_j}$ is LSC by Prop. 7.11 (5), $g \geq f$, and $\int g\,d\mu \leq \int f\,d\mu + \varepsilon$.

Let $\alpha < \int f\,d\mu$. $\exists N$ s.t. $\sum_{j=1}^N a_j \mu(E_j) > \alpha$. E_j is σ-finite,

$\Longrightarrow \mu$ is regular at $E_j \Longrightarrow \exists$ compact $K_j \subseteq E_j$ s.t. $\sum_1^N a_j \mu(K_j) > \alpha$.

Let $g = \sum_1^N a_j \chi_{K_j}$, then g is USC, $g \leq f$ and $\int g\,d\mu > \alpha$.

QED