Monday, 4/13/2020, Lecture 7

§ 7.3 The Dual of $C_0(X)$.

Ideas X: a LCH space.

⊙ $C_0(X)$ = the closure of $C_c(X)$ w.r.t. the uniform norm: $\|f\| = \sup_{x\in X}|f(x)| = \max_{x\in X}|f(x)|$.

⊙ μ: Radon meas. on X. $I(f) = \int f\,d\mu \quad \forall f \in C_c(X)$.

$\Longrightarrow$ I is positive + linear on $C_c(X)$

If I is bounded. $\exists C>0$. s.t. $|I(f)| \le C\|f\| \quad \forall f \in C_c(X)$

$\Longrightarrow$ Extend I to $C_0(X)$. So, $I \in C_0(X)^*$.

⊙ Q: When I is bounded ?

$\mu(X) = \sup\{\int f\,d\mu : f \in C_c(X, [0,1])\}$.

I is bounded $\Longleftrightarrow \mu(X) < \infty. \Longrightarrow \|I\| = \mu(X)$.

Conclusion: $I \in C_0(X)^*$ + positive $\Longleftrightarrow$ finite Radon meas. μ.

⊙ General $I \in C_0(X)^*$: $I = I^+ - I^-$. Jordan decomp.

Main Results

Def. X: LCH.
$M(X) = \{$all complex Radon meas. on $X\}$.

$\mu \in M(X)$: $\|\mu\| = |\mu|(X)$. $I_\mu(f) = \int_X f\,d\mu \quad \forall f \in C_0(X)$

<u>The Riesz Rep. Thm</u> X: LCH. The map $\mu \longmapsto I_\mu$ is an isometric isomorphism between $M(X)$ and $C_0(X)^*$.

Remarks

- Notation $C_0(X)^* \cong M(X)$.
- A special case: X is compact + Hausdorff

$\Longrightarrow C(X) = C_0(X) = C_c(X) \Longrightarrow C(X)^* \cong M(X)$.

- $\forall \mu \in M(X)$, $\mu \geq 0$. Let $f \in L^1(\mu)$. Def.

$\nu_f(E) = \int_E f\,d\mu$. Then, $\nu_f \in M(X)$, $\|\nu_f\| = \int |f|\,d\mu$.

Thus. $L^1(\mu) \subseteq M(X)$.

<u>Lemma 7.15</u> X: LCH, $I \in C_0(X,\mathbb{R})^* \Longrightarrow \exists I^\pm \in C_0(X,\mathbb{R})^*$ positive, s.t. $I = I^+ - I^-$ (Jordan decomposition).

<u>Pf</u> $\forall f \in C_0(X,[0,\infty))$, def.

$$I^+(f) = \sup\{I(g) : g \in C_0(X,\mathbb{R}),\ 0 \le g \le f\}.$$

[Next: $\forall f \in C_0(X,\mathbb{R})$, $f = f^+ - f^-$. $I^+(f) = I^+(f^+) - I^+(f^-)$. $I^- = I^+ - I$]

$\odot$ $0 \le g \le f \Longrightarrow |I(g)| \le \|I\|\,\|g\| \le \|I\|\,\|f\| \Longrightarrow \underline{0 \le I^+(f) \le \|I\|\,\|f\|}$.

$\odot$ $\underline{I^+(cf) = cI^+(f)}$ $\forall c \ge 0$, $\forall f \in C_0(X,[0,\infty))$ [check by def.]

$\odot$ $f_1, f_2 \in C_0(X,[0,\infty)) \Longrightarrow \underline{I^+(f_1+f_2) = I^+(f_1) + I^+(f_2)}$.

<u>Pf</u> $0 \le g_1 \le f_1$, $0 \le g_2 \le f_2 \Longrightarrow 0 \le g_1 + g_2 \le f_1 + f_2$.

$\Longrightarrow I^+(f_1+f_2) \ge I(g_1+g_2) = I(g_1) + I(g_2) \Longrightarrow \underline{I^+(f_1+f_2) \ge I^+(f_1) + I^+(f_2)}$.

$\forall g: 0 \le g \le f_1 + f_2$. Let $g_1 = \min(g, f_1)$, $g_2 = g - g_1 \Rightarrow g = g_1 + g_2$

$0 \le g_1 \le f_1$, $0 \le g_2 \le f_2 \Longrightarrow I(g) = I(g_1) + I(g_2) \le I^+(f_1) + I^+(f_2)$

$\Longrightarrow \underline{I^+(f_1+f_2) \le I^+(f_1) + I^+(f_2)}$.

$\forall f \in C_0(X,\mathbb{R})$: $f^{\pm} = \max(\pm f, 0) \in C_0(X,[0,\infty))$, $f = f^+ - f^-$.

Def. $I^+(f) = I^+(f^+) - I^+(f^-)$. If $f = g - h$, $g, h \in C_0(X,[0,\infty))$,

then $g + f^- = h + f^+ \Longrightarrow I^+(g) + I^+(f^-) = I^+(h) + I^+(f^+)$

$\Longrightarrow I^+(g) - I^+(h) = I^+(f^+) - I^+(f^-) = I^+(f)$

⊙ I^+ is linear on $C_0(X,\mathbb{R})$.

$a \in \mathbb{R}$, $f \in C_0(X,\mathbb{R}) \Longrightarrow I^+(af) = aI^+(f)$. [Consider $a \begin{smallmatrix} >0 \\ =0 \\ <0 \end{smallmatrix}$]

$f_1, f_2 \in C_0(X,\mathbb{R}) \Longrightarrow I^+(f_1+f_2) = I^+(f_1 + f_2)$.

<u>Pf</u> $f \triangleq f_1 + f_2 \Rightarrow f^+ - f^- = f_1^+ - f_1^- + f_2^+ - f_2^-$

$\Longrightarrow f^+ + f_1^- + f_2^- = f_1^+ + f_2^+ + f^- \Longrightarrow$

$I^+(f) = I^+(f^+) - I^+(f^-) = I^+(f_1^+) + I^+(f_2^+) - I^+(f_1^-) - I^+(f_2^-) = I^+(f_1) + I^+(f_2)$.

⊙ I^+ is bounded on $C_0(X,\mathbb{R})^*$, $\|I^+\| \le \|2I\|$: $\forall f \in C_0(X,\mathbb{R})$:

$|I^+(f)| \le \max(I^+(f^+), I^+(f^-)) \le \|I\| \max(\|f^+\|, \|f^-\|) = \|I\|\,\|f\|$.

Finally, def $I^- = I^+ - I \in C_0(X,\mathbb{R})^*$, $I^- \ge 0$, $I = I^+ - I^-$. <u>QED</u>

Def. ① A signed Borel meas. is Radon if its pos. + neg. variations are Radon.

② A complex Borel meas. is Radon if its real + imaginary parts are Radon

Conseq. of Lemma

- $I \in C_0(X, \mathbb{R})^* \iff \exists \mu_1, \mu_2$: Radon s.t.

$$I(f) = I^+(f) - I^-(f) = \int f\,d\mu_1 - \int f\,d\mu_2 \quad \forall f \in C_c(X, \mathbb{R}).$$

- $I \in C_0(X)^*$ is uniquely determined by $J = I|_{C_0(X,\mathbb{R})} = J_1 + iJ_2$, $J_1, J_2 \in C_0(X,\mathbb{R})^*$. $\Longrightarrow \exists$ finite Radon meas. μ_j $(1 \le j \le 4)$ s.t.

$$I(f) = \int f\,d\mu \quad \forall f \in C_0(X), \text{ where } \mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4).$$

Prop 7.16 X : LCH.

- μ : Borel on X : μ is Radon $\iff |\mu|$ is Radon.
- $(M(X), \|\cdot\|)$ is a normed vector space.

Pf Exercise. **QED** $J \triangleq I|_{C_0(X,\mathbb{R})} \in C_0(X,\mathbb{R})^*$

The Riesz Rep. Thm: X: LCH. $C_0(X)^* \cong M(X)$ $\boxed{35}$
under $\mu \longmapsto I_\mu$. [In particular, $M(X)$ is Banach.]

Pf $I \in C_0(X)^* \Longrightarrow I = I_\mu$ for some $\mu \in M(X)$.

$\forall \mu \in M(X)$. $|I_\mu(f)| = \left|\int f\, d\mu\right| \le \int |f|\, d|\mu| \le \|f\|\,\|\mu\|$.

So, $I_\mu \in C_0(X)^*$ and $\underline{\|I_\mu\| \le \|\mu\|}$.

Let $h = d\mu / d|\mu|$. The $|h| = 1$. By Lusin's Thm, $\forall \varepsilon > 0$. $\exists f \in C_c(X)$ s.t. $\|f\| = 1$ and $f = \bar{h}$ on $E \in \mathcal{B}_X$, $|\mu|(E^c) < \varepsilon/2$. Thus,

$$\|\mu\| = \int |h|^2\, d|\mu| = \int \bar{h}\, d\mu \le \left|\int f\, d\mu\right| + \left|\int (f - \bar{h})\, d\mu\right|$$

$$\le \|I_\mu\|\,\|f\| + 2|\mu|(E^c) \le \|I_\mu\| + \varepsilon \Longrightarrow \underline{\|\mu\| \le \|I_\mu\|}.$$

Thus, $\|I_\mu\| = \|\mu\|$. QED

Def. X: LCH. $C_0(X)^* \cong M(X)$. The *vague topology* on $M(X)$ is the weak-$*$ topology on $C_0(X)^*$, defined by $C_0(X)$.

$\mu_\alpha \to \mu$ in $M(X)$ in vague topology

$\iff \int f\, d\mu_\alpha \to \int f\, d\mu \quad \forall f \in C_0(X)$.

Remarks

- $\mu_n \to \mu$ in $M(X) \Rightarrow \mu_n \to \mu$ vaguely.
- $\mu_n \to \mu$ vagulely $\not\Rightarrow \mu_n(E) \to \mu(E) \; \forall E \in \mathcal{B}_X$.