Wednesday, 4/15/2020. Lecture 8

§7.4 Products of Radon Measures

Consider: X, Y: LCH spaces, μ, ν: Radon meas. on X, Y.

Questions/objectives:

$\mu \times \nu$: Radon meas. on $X \times Y$? construction?

Integral $\int f \, d(\mu \times \nu)$, Fubini-Tonelli Thm.

Review $X \times Y = \{(x, y): x \in X, y \in Y\}$,

π_X, π_Y: projections: $\pi_X: X \times Y \to X$, $\pi_X(x, y) = x$,

$\pi_Y: X \times Y \to Y$, $\pi_Y(x, y) = y$.

$\mathcal{T}_X, \mathcal{T}_Y$: topologies of X, Y. The product topology $\mathcal{T}_{X \times Y}$ is generated by $\pi^{-1}(U) \times \pi^{-1}(V)$, $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$.

(i) π_X, π_Y: continuous.

(ii) $X \times Y$ is also LCH.

$\mathcal{B}_X, \mathcal{B}_Y$: Borel σ-alg. on X, Y. $\mathcal{B}_X \otimes \mathcal{B}_Y$: product σ-alg., generated by $\{A \times B : A \in \mathcal{B}_X, B \in \mathcal{B}_Y\}$.

Theorem 7.20 X, Y: LCH spaces.

① $\mathcal{B}_X \times \mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}$.

② X, Y: second countable $\Longrightarrow \mathcal{B}_X \times \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.

③ X, Y: second countable, μ, ν: Radon measures on X, Y, resp. $\Longrightarrow \mu\times\nu$ is Radon on $X\times Y$.

Recall: $\mu\times\nu$ is the unique measure on $X\times Y$ s.t.

$$(\mu\times\nu)(A\times B) = \mu(A)\nu(B), \quad A\in\mathcal{B}_X,\ B\in\mathcal{B}_Y.$$

Pf ① $\mathcal{T}_X, \mathcal{T}_Y$ generate $\mathcal{B}_X, \mathcal{B}_Y$. By Prop. 1.4, $\mathcal{B}_X\otimes\mathcal{B}_Y$ is generated by $\mathcal{T}_X\times\mathcal{T}_Y \triangleq \{U\times V : U\in\mathcal{T}_X, V\in\mathcal{T}_Y\}$, which is $\subseteq \mathcal{T}_{X\times Y}$. Since $\mathcal{B}_{X\times Y}$ is generated by $\mathcal{T}_{X\times Y}$, $\mathcal{B}_X\otimes\mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}$.

② Let $\mathcal{E}, \mathcal{F}$ be countable bases for X, Y, resp. Denote $\mathcal{E}\times\mathcal{F} \triangleq \{U\times V : U\in\mathcal{E}, V\in\mathcal{F}\}$ and Γ the σ-alg. on $X\times Y$ generated by $\mathcal{E}\times\mathcal{F}$. Since $\mathcal{E}\times\mathcal{F}\subseteq\mathcal{B}_X\otimes\mathcal{B}_Y$, $\Gamma\subseteq\mathcal{B}_X\otimes\mathcal{B}_Y$. Since any member in $\mathcal{T}_{X\times Y}$ is a countable union of members in $\mathcal{E}\times\mathcal{F}$, $\mathcal{T}_{X\times Y}\subseteq\Gamma$. So, $\mathcal{B}_{X\times Y}\subseteq\Gamma\subseteq\mathcal{B}_X\otimes\mathcal{B}_Y$, all $=$ by ①.

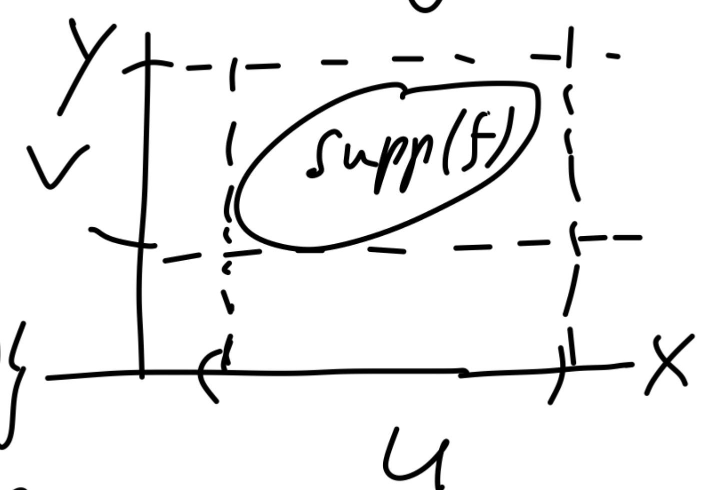
③ $\mu \times \nu$ is a Borel meas on $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.
$X \times Y$ is also second countable. So, by Thm 7.8, it suffices to show that $\mu\times\nu(K) < \infty$ if K is compact in $X \times Y$. In this case, $\pi_X(K)$, $\pi_Y(K)$ are compact in X, Y, resp. $K \subseteq \pi_X(K) \times \pi_Y(K)$.
Thus, $\mu\times\nu(K) \le \mu\times\nu(\pi_X(K)\times\pi_Y(K))$
$$= \mu(\pi_X(K))\, \nu(\pi_Y(K)) < \infty.$$ QED

Remark ① X, Y: not second countable, then $\mathcal{B}_X \otimes \mathcal{B}_Y \ne \mathcal{B}_{X\times Y}$ possible! $\mu \times \nu$: not Radon.
② Ideas: construct product Radon measures through integrals.

Def. $g: X \to \mathbb{C}$, $h: Y \to \mathbb{C}$: $(g\otimes h)(x,y) = g(x)\, h(y)$.

Stone-Weierstrass Thm If X_0, Y_0 are compact + Hausdorff, then $\operatorname{span}\{ g\times h : g \in C(X_0),\ h \in C(Y_0)\}$ is dense in $C(X_0 \times Y_0)$.

Prop 7.21 Let $\mathcal{P} = \text{span}\{g \otimes h : g \in C_c(X), h \in C_c(Y)\}$.
Then $\mathcal{P}$ is dense in $C_c(X \times Y)$. More precisely, $\forall f \in C_c(X \times Y)$, $\forall \varepsilon > 0$, $\forall$ precompact open $U \subseteq X$, $V \subseteq Y$ s.t.
$\pi_X(\text{supp}(f)) \subseteq U$, $\pi_Y(\text{supp}(f)) \subseteq V$ $\Longrightarrow$ $\exists F \in \mathcal{P}$ with $\|F - f\| < \varepsilon$ and $\text{supp}(F) \subseteq U \times V$.



Pf $\bar{U} \times \bar{V}$ is compact Hausdorff.
S-W Thm $\Rightarrow$ $\text{Span}\{g \otimes h : g \in C(\bar{U}), h \in C(\bar{V})\}$ is dense in $C(\bar{U} \times \bar{V})$ $\Longrightarrow$ $\exists$ G in this span, s.t. $\sup_{\bar{U} \times \bar{V}} |G - f| < \varepsilon$. Now, Urysohn's $\Longrightarrow$ $\exists \phi \in C_c(U, [0,1])$, $\exists \psi \in C_c(V, [0,1])$ s.t. $\pi(\text{supp}(f)) \prec \phi \prec U$, $\pi(\text{supp}(f)) \prec \psi \prec V$.
Def $F = (\phi \otimes \psi) G$ on $\bar{U} \times \bar{V}$ and $F = 0$ elsewhere. Then $F \in \mathcal{P}$, $\text{supp}(F) \subseteq U \times V$, and $\|F - f\| < \varepsilon$.
(On $\text{supp}(f)$, $\phi \otimes \psi(x,y) = \phi(x)\psi(y) = 1$. On $\bar{U} \times \bar{V} \setminus \text{supp}(f)$, $f = 0$, $|\phi \otimes \psi| \leq 1$. $\sup |G| < \varepsilon$.)

QED

<u>Prop. 7.22</u> Every $f \in C_c(X \times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable. Moreover, if μ, ν are Radon measures on X and Y, resp., then $C_c(X \times Y) \subseteq L^1(\mu \times \nu)$, and

$$\int f \, d(\mu \times \nu) = \iint f \, d\mu \, d\nu = \iint f \, d\nu \, d\mu.$$

<u>Pf</u> $g \in C_c(X)$, $h \in C_c(Y) \Longrightarrow g \otimes h = (g \circ \pi_X)(h \circ \pi_Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$-measurable, since π_X, π_Y are measurable from $\mathcal{B}_X \otimes \mathcal{B}_Y$ to $\mathcal{B}_X$, $\mathcal{B}_Y$, and g, h are continuous. Hence, any $f \in C_c(X \times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$-meas. by Prop. 7.21.

Now, $f \in C_c(X \times Y) \Longrightarrow f$ is bounded, supported in a set of finite $\mu \times \nu$-measure. hence, $f \in L^1(\mu \times \nu)$. Fubini's Thm holds for such f with the finite measures $\mu|_{\pi_X(\mathrm{supp}(f))}$ and $\nu|_{\pi_Y(\mathrm{supp}(f))}$, resp. <u>QED</u>

<u>Def</u> X, Y: LCH, μ, ν: Radon on X, Y. The Radon product meas. $\mu \hat{\times} \nu$ is the Radon meas on $X \times Y$ associated to the functional $f \mapsto \int f \, d(\mu \times \nu)$, $\forall f \in C_c(X \times Y)$.