Friday, 4/17/2020, Lecture 9

§7.4 Products of Radon Measures (cont'd)

Review X, Y: LCH, μ, ν: Radon measures on X, Y.

$I(f) = \int f \, d\mu\times\nu$ def. linear + pos. $I: C_c(X\times Y) \to \mathbb{C}$.

$\Longrightarrow \exists!$ Radon measure on $(X\times Y, \mathcal{B}_{X\times Y})$, $\mu \hat{\times} \nu$,

the product Radon meas. of μ and ν. s.t.

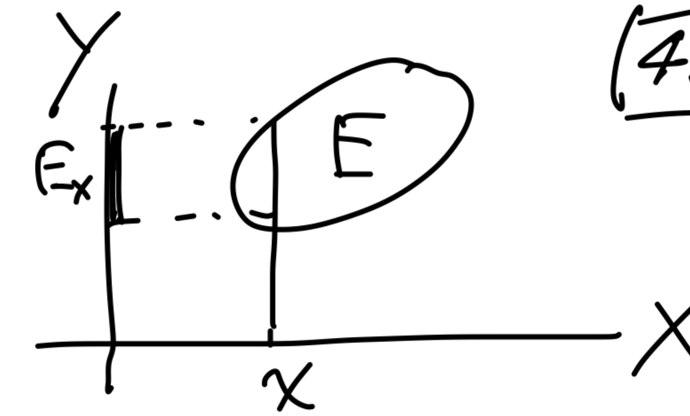
$$\int f \, d\mu\hat{\times}\nu = I(f) = \int f \, d\mu\times\nu \quad \forall f \in C_c(X\times Y).$$

Thm 7.26 X, Y: LCH, μ, ν: σ-finite Radon measures on X, Y, resp. $E \in \mathcal{B}_{X\times Y}$. Then:

- $E_x \in \mathcal{B}_Y$ $(\forall x\in X)$, $E^y \in \mathcal{B}_X$ $(\forall y \in Y)$;
- $x \mapsto \nu(E_x)$, $y \mapsto \mu(E^y)$: Borel-meas. on X, Y resp.;
- $\mu\hat{\times}\nu(E) = \int \nu(E_x) \, d\mu(x) = \int \mu(E^y) \, d\nu(y)$.

Moreover, $\mu\hat{\times}\nu|_{\mathcal{B}_X\otimes\mathcal{B}_Y} = \mu\times\nu$.

Recall. ⊙ $\forall E \subseteq X\times Y$, $\forall x \in X$, $\forall y \in Y$:



$E_x = \{y \in Y : (x,y) \in E\} \subseteq Y$.

$E^y = \{x \in X : (x,y) \in E\} \subseteq X$.

Fact. E open $\Longrightarrow E_x, E_y$ open.

⊙ $f : X\times Y \to \mathbb{C}$, $x \in X$, $y \in Y$: $f_x : Y \to \mathbb{C}$. $f^y : X \to \mathbb{C}$. $f_x(y) = f_y(x) = f(x,y)$

Fact: $S \subseteq \mathbb{C} \Longrightarrow f_x^{-1}(S) = (f^{-1}(S))_x$ $(f^y)^{-1}(S) = (f^{-1}(S))^y$.

<u>Plan for Proof:</u> First, E is open. Then, general E.

<u>Lemma 7.23</u> (1) $E \in \mathcal{B}_{X\times Y} \Longrightarrow E_x \in \mathcal{B}_Y$ $(\forall x \in X)$, $E^y \in \mathcal{B}_X$ $(\forall y \in Y)$.

(2) $f : X\times Y \to \mathbb{C}$ is $\mathcal{B}_{X\times Y}$-measurable $\Longrightarrow f_x$ is $\mathcal{B}_Y$-meas. $(\forall x \in X)$ and f^y is $\mathcal{B}_X$-meas. $(\forall y \in Y)$

<u>Pf</u> (1) $\mathcal{A} \triangleq \{A \subseteq X\times Y : A_x \in \mathcal{B}_Y\ \forall y \in Y,\ A^y \in \mathcal{B}_X\ \forall x \in X\}$ is a σ-alg.

A open $\Longrightarrow A_x, A^y$ open $\Longrightarrow \mathcal{A} \supseteq \mathcal{T}_{X\times Y} \Longrightarrow \mathcal{A} \supseteq \mathcal{B}_{X\times Y} \ni E \Longrightarrow E \in \mathcal{A}$.

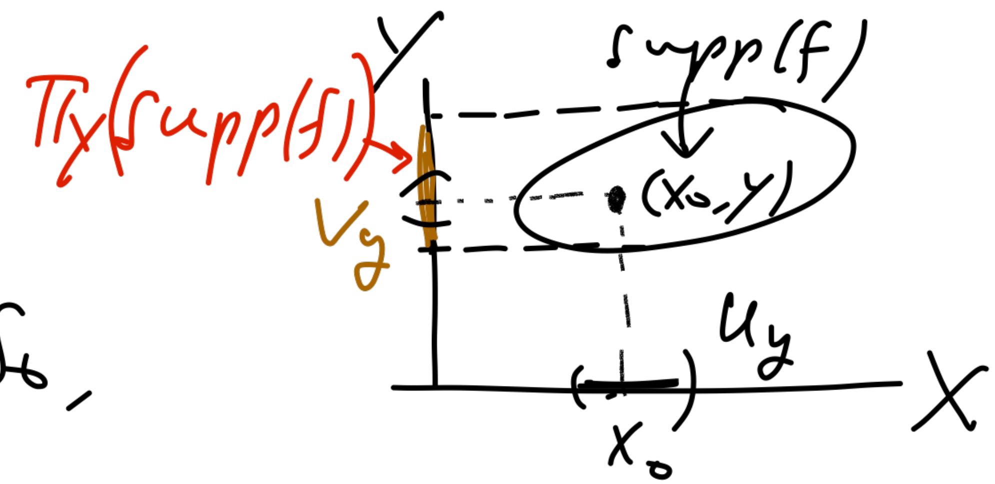
(2) $S \subseteq \mathbb{C}$: $f_x^{-1}(S) = (f^{-1}(S))_x$. $(f^y)^{-1}(S) = (f^{-1}(S))^y$. Use (1). <u>QED</u>

$\longleftarrow \in \mathcal{B}_{X\times Y} \longrightarrow$

Lemma 7.24 $f \in C_c(X \times Y)$, μ, ν : Radon on X, Y

$\Longrightarrow x \mapsto \int f_x \, d\nu$, $y \mapsto \int f^y \, d\mu$ are continuous.

Pf $\forall x_0 \in X$, $\forall \varepsilon > 0$. Show $\exists$ open nbh U of x_0 s.t. $\|f_x - f_{x_0}\| < \varepsilon$ $(\forall x \in U)$.



If this is done, then $\forall x \in U$:

$\left| \int (f_x - f_{x_0}) \, d\nu \right| \leq \varepsilon \, \nu(\Pi_Y(\mathrm{supp}(f)))$. So, $x \mapsto \int f_x \, d\nu$ is continuous.

Find U $\forall y \in \Pi_Y(\mathrm{supp}(f))$. f is cont. at $(x_0, y) \Rightarrow \exists$ open U_y, V_y s.t. $x_0 \in U_y \subseteq X$, $y \in V_y \subseteq Y$, and $\forall (x, z) \in U_y \times V_y$, $|f(x,z) - f(x_0, y)| < \varepsilon$. Since all such V_y cover $\Pi_Y(\mathrm{supp}(f))$ which is compact as $\mathrm{supp}(f)$ is and Π_Y is cont. $\exists V_{y_1}, \dots, V_{y_m}$ covering $\Pi_Y(\mathrm{supp}(f))$. Now let $U = \bigcap_{j=1}^{m} U_{y_j}$ open, $x_0 \in U$. If $x \in U$ and $y \in \Pi_Y(\mathrm{supp}(f))$, then $y \in V_{y_j}$ for some j. $|f_x(y) - f_{x_0}(y)| = |f(x, y) - f(x_0, y)| < \varepsilon$. Hence, $\|f_x - f_{x_0}\| < \varepsilon$. QED

Prop. 7.25 X, Y: LCH, μ, ν: Radon meas. on X, Y. U open in $X\times Y$. Then $x \mapsto \nu(U_x)$, $y \mapsto \mu(U^y)$ are Borel measurable on X and Y resp., and

$$\mu \hat{\times} \nu(U) = \int \nu(U_x)\,d\mu(x) = \int \mu(U^y)\,d\nu(y).$$

Pf let $\mathcal{F} = \{f \in C_c(X\times Y) : 0 \le f \le \chi_U\}$. Prop. 7.11 $\Rightarrow$ nonneg. LSC
$\chi_U = \sup\{f : f \in \mathcal{F}\} \Longrightarrow \chi_{U_x} = (\chi_U)_x = \sup\{f_x : f \in \mathcal{F}\}$.
$\chi_{U^y} = \sup\{f^y : f \in \mathcal{F}\}$. Prop. 7.12 $\Longrightarrow (\mu\hat{\times}\nu)(U) = \int \chi_U \, d\mu\hat{\times}\nu$
$= \sup\{\int f\, d(\mu\hat{\times}\nu) : f \in \mathcal{F}\}$. $\nu(U_x) = \int \chi_{U_x} d\nu = \sup\{\int f_x\, d\nu : f \in \mathcal{F}\}$,
$\mu(U^y) = \int \chi_{U^y} d\mu = \sup\{\int f^y d\mu : f \in \mathcal{F}\}$.

Now, Lemma 7.24 + Prop. 7.11 $\Longrightarrow x \mapsto \nu(U_x)$, $y \mapsto \mu(U^y)$ are LSC and hence Borel measurable. Prop 7.12 + 7.22

$$\Longrightarrow \mu\hat{\times}\nu(U) \overset{(7.22)}{=} \sup\{\int f\, d\mu\times\nu : f \in \mathcal{F}\} \overset{(old)}{=} \sup\{\int\int f_x\, d\nu\, d\mu(x) : f \in \mathcal{F}\}$$

$$\overset{(7.12)}{=} \int \left[\sup\{\int f_x\, d\nu : f \in \mathcal{F}\}\right] d\mu(x) = \int \nu(U_x)\, d\mu(x).$$

Similarly, $\mu\hat{\times}\nu(U) = \int \mu(U^y)\, d\nu(y)$. QED