Problem #1

If \( x \in K \) then \( f(x) = K^c(x) = 1 \). If \( x \in K^c \) then \( f(x) = 0 \) and \( K^c(x) = 0 \). (Hence \( f \geq K^c \).

Now, \( M(K) = \int 1 \, \text{d}u = \int_{K} f \, \text{d}u \leq \int_{U} f \, \text{d}u \). Since \( K \subseteq U, f \geq 0 \),

Since \( f \leq 1 \), \( \int_{U} 1 \, \text{d}u = M(U) \).
Problem #2

Assume \( \mu \) is a Radon measure. Since \( \mu \) is finite, it is \( \sigma \)-finite. Thus, by Proposition 7.7: \( \forall E \in \mathcal{B}, \forall \varepsilon > 0, \exists K: \text{compact}, U: \text{open}, \text{s.t.} \ K \subseteq E \subseteq U \text{ and } \mu(U \setminus K) < \varepsilon. \)

Suppose for any \( E \in \mathcal{B} \) and any \( \varepsilon > 0 \) there exist \( K: \text{compact} \) and \( U: \text{open}, \text{s.t.} \ K \subseteq E \subseteq U \text{ and } \mu(U \setminus K) < \varepsilon. \) Then, \( \mu(K) = \mu(E) - \varepsilon \) (as \( \mu(E \setminus K) \leq \mu(U \setminus K) < \varepsilon \)), and \( \mu(U) = \mu(E) + \varepsilon \) (as \( \mu(U \setminus E) \leq \mu(U \setminus K) < \varepsilon \)).

Thus \( \mu \) is regular. Since \( \mu \) is finite, it is finite on compact sets. Thus \( \mu \) is a Radon measure.
Problem #3

(1) True. If $\mu$ is a finite Borel measure on $\mathbb{R}^n$ then it is finite on compact sets. If $U$ is open in $\mathbb{R}^n$ then it is the countable union of compact sets $B(x,r)$ where $x \in U \cap \mathbb{Q}^n$ and $r \in (0,\infty) \cap \mathbb{Q}$. Hence, by Thm 7.8, $\mu$ is a Radon measure.

(2) False. Example. $X = \mathbb{R}^n$, $\mu =$ Lebesgue measure. $f(x) = \frac{1}{1+|x|}$, $x \in \mathbb{R}$. $f \in C_c(\mathbb{R})$ but $f \not\in L^1(\mathbb{R})$. 
Problem 4

Let $U \subseteq X$ be open. Let $\varepsilon > 0$. By the Kiesz representative, then for positive and linear functionals on $C_c(X)$:

$$
\mu(U) = \sup \left\{ \int f \, d\mu : f \in C_c(X), f \leq U \right\}.
$$

Since $\mu$ is finite, $\exists f \in C_c(X), f \leq U$ s.t.

$$
\mu(U) \leq \int f \, d\mu + \varepsilon. \quad \text{Since } \mu_n \to \mu \text{ vaguely,}
$$

$$
\int f \, d\mu_n \to \int f \, d\mu. \quad \text{Hence, } \mu(U) \leq \lim_{n \to \infty} \int f \, d\mu_n + \varepsilon.
$$

But each $\int f \, d\mu_n \leq \mu_n(U)$ ($n = 1, 2, \ldots$).

Then, $\lim_{n \to \infty} \int f \, d\mu_n = \lim_{n \to \infty} \int f \, d\mu_n \leq \lim_{n \to \infty} \mu_n(U)$

i.e., $\mu(U) \leq \lim_{n \to \infty} \mu_n(U) + \varepsilon$

Thus $\mu(U) \leq \lim_{n \to \infty} \mu_n(U) \leq \lim_{n \to \infty} \sup \mu_n(U)$. 

Let \( F \subseteq X \) be compact. Then by the Riesz Representation Theorem for positive linear functionals,

\[
\mu(F) = \inf \{ \int f \, d\mu : f \in C(X), f \geq \chi_F \}.
\]

\( \forall \varepsilon > 0 \). Since \( \mu(F) < \infty \) (as \( \mu \) is a Radon measure), \( \exists f \in C_c(X), f \geq \chi_F \) s.t.

\[
\mu(F) > \int f \, d\mu - \varepsilon. \quad \text{Thus, since } \|f\|_{C_c} \to \|f\|_{C_c},
\]

\[
\mu(F) > \lim_{n \to \infty} \int f \, d\mu_n - \varepsilon = \limsup_{n \to \infty} \int f \, d\mu_n - \varepsilon \geq \limsup_{n \to \infty} \mu_n(F) - \varepsilon.
\]

Thus, \( \liminf_{n \to \infty} \mu_n(F) \leq \limsup_{n \to \infty} \mu_n(F) \leq \mu(F) \).