Math 240C: Real Analysis, Spring 2020

Homework Assignment 1
Due 12:00 noon, Monday, April 6, 2020

1. Let $X$ be a locally compact Hausdorff (LCH) space, $Y$ a closed subset of $X$ (which is an LCH space in the relative topology), and $\mu$ a Radon measure on $Y$. Define $I : C_c(X) \to \mathbb{C}$ by

$$I(f) = \int_Y (f|_Y) \, d\mu \quad \forall f \in C_c(X),$$

where $f|_Y$ is the restriction of $f$ onto $Y$. Prove that $I$ is a positive linear functional on $C_c(X)$ and that the induced Radon measure $\nu$ on $X$ is given by $\nu(E) = \mu(E \cap Y)$ for any Borel measurable subset $E$ of $X$.

2. Let $X$ be a locally compact Hausdorff space and $I$ a positive linear functional on $C_c(X)$. Prove that for any compact subset $K$ of $X$ there exists $C_K \in \mathbb{R}$, depending on $K$, such that $|I(f)| \leq C_K \|f\|$ for any $f \in C_c(X)$ such that $\text{supp}(f) \subseteq K$.

3. Let $\mu$ be a Radon measure on a locally compact Hausdorff space $X$.
   (1) Let $N$ be the union of all open subsets $U \subseteq X$ such that $\mu(U) = 0$. Prove that $N$ is the largest open subset of $X$ such that $\mu(N) = 0$. The complement of $N$ is called the support of $\mu$ and is denoted by $\text{supp}(\mu)$.
   (2) Prove that $x \in \text{supp}(\mu)$ if and only if

   $$\int_X f \, d\mu > 0 \quad \text{for all } f \in C_c(X, [0, 1]) \text{ such that } f(x) > 0.$$ 

4. Let $\mu$ be a Radon measure on a locally compact Hausdorff space $X$ and $\phi \in L^1(\mu)$ with $\phi \geq 0$ on $X$. Define

$$\nu(E) = \int_E \phi \, d\mu \quad \forall E \in \mathcal{B}_X.$$ 

Prove that $\nu$ is a Radon measure on $X$ and that $\text{supp}(\nu) \subseteq \text{supp}(\phi) \cap \text{supp}(\mu)$.

5. Let $X$ be a locally compact Hausdorff space and $x_0 \in X$. Define $I(f) = f(x_0)$ for any $f \in C_c(X)$. Prove that $I : C_c(X) \to \mathbb{C}$ is a positive linear functional on $C_c(X)$ and that the Radon measure associated with the functional $I$ is the Dirac mass $\delta_{x_0}$ at $x_0$.

6. Let $\mu$ be a Radon measure on a compact Hausdorff space $X$ with $\mu(X) = 1$. Prove that there is a compact set $K \subseteq X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset $H$ of $K$.

7. Prove that a Borel measure on $\mathbb{R}^n$ is a Radon measure if and only if it is finite on each compact subset of $\mathbb{R}^n$. 