1. If \( f \in C_0(X) \) and \( f \geq 0 \) on \( X \), then \( f \upharpoonright Y \geq 0 \). Hence \( I(f) = \int_Y f \upharpoonright Y \, du \geq 0 \). Thus \( I \) is positive.

Since the restrictions \( f \upharpoonright Y = X \upharpoonright f \) and the integration are both linear, \( I \) is linear.

Verify that \( V \) is a Radon measure on \( X \).

1. Let \( K \subseteq X \) be compact. So, \( K \) is closed, and \( K \upharpoonright Y \) is closed. \( K \upharpoonright Y \subseteq Y \). So, \( K \upharpoonright Y \) is compact in \( X \).

Let \( \{U_0, \cup \} \) be an open cover of \( K \upharpoon \right Y \) in \( Y \), where \( U_0 \) is open in \( X \). Then \( U_0(x(A)) \subseteq Y \) forms an open cover of \( K \upharpoon \right Y \) in \( X \). So, \( \exists \{U_1, \cup \} \) s.t. \( U_1, \cup \subseteq Y \) cover \( K \upharpoon \right Y \) in \( X \). Hence \( \{U_1, \cup \} \) covers \( K \upharpoon \right Y \), and \( K \upharpoon \right Y \) is compact in \( Y \). Thus \( V(K) = \int (K \upharpoon \right Y) \).
(2) If $E$ is a Borel subset of $X$, then $E \cap Y$ is a Borel subset of $Y$. Since $\mu$ is a Radon measure on $Y$, it is outer regular at $E \cap Y$, i.e.,

$$\mu(E \cap Y) = \inf \{ \mu(U \cap Y) : U \text{ open in } X, U \cap Y \supseteq E \cap Y \}$$

(Open sets in $Y$ are of the form $U \cap Y$, $U$ open in $X$.)

Thus, by the def. of $V$,

$$V(E) = \inf \{ V(U) : U \text{ open in } X, U \cap Y \supseteq E \cap Y \} =: a.$$

Let $b = \inf \{ V(U) : U \text{ open in } X, U \supseteq E \}$. We have $a \leq b$ since $U \supseteq E \Rightarrow U \cap Y \supseteq E \cap Y$. If $U$ is open in $X$ and $U \cap Y \supseteq E \cap Y$ then for $V = U \cap Y$ open in $X$, $V \supseteq E$, and $V(V) = \mu(V \cap Y) = \mu(U \cap Y) = \mu(U)$.

Thus $a \geq b$. Hence $a = b$, and $V$ is outer regular.

(3) Let $U$ be open in $X$. Then $U \cap Y$ is open in $Y$. Since $\mu$ is inner regular at $U \cap Y$,
\[ \nu(U) = \mu(U \cap Y) \]
\[ = \inf \{ \mu(F) : F \text{ compact in } Y, F \subseteq U \} \]
\[ = \inf \{ \mu(F \cap Y) : F \text{ compact in } Y, F \subseteq U \} \]
\[ = \inf \{ \mu(K \cap Y) : K \text{ compact in } X, K \subseteq U \}. \]

Denote \( d = \inf \{ \nu(K) : K \text{ compact in } X, K \subseteq U \}. \)

If \( F \) is compact in \( Y \) and \( F \subseteq U \), then \( F \) is also compact in \( X \) as \( Y \) is closed in \( X \). Thus, \( d \leq c \).

Let \( K \) be compact in \( X \) and \( K \subseteq U \). Let \( F = K \cap Y \).

Then, \( F \) is compact in \( Y \) (cf. Part 1) and \( F \subseteq U \).

Moreover, \( \mu(K \cap Y) = \mu(F \cap Y) \geq c \). Hence, \( d \geq c \).

Thus, \( d = c \), and \( \nu \) is inner regular at \( U \).

By (a) - (b), \( \nu \) is a Radon measure on \( X \). Now, \( \forall f \in C_c(X), \int f \, d\nu = \int f \, d\nu + \int f \, d\nu = \int f (Y \cup X) \, d\nu = I(f) \).

By the uniqueness, \( \nu \) is exactly the Radon measure associated with \( I \).
2. Let \( K \subseteq X \) be compact. Let \( f_K \in C_c(X, [0, 1]) \) be such that \( K \subseteq f_K^c \). Let \( f \in C_c(X) \) with \( \text{supp}(f) \subseteq K \). Then \( \|f\|_1 f_K \leq f \leq f_K \) on \( X \). So, \( I(\|f\|_1 f_K \pm f) \geq 0 \), i.e., \( \|f\|_1 I(\phi_K) \pm I(f) \geq 0 \). Hence, \( |I(f)| \leq I(\phi) \|f\|_1 \). Set \( \zeta_K = I(\phi_K) \).

3. (1) Clearly \( N \) is open. Since \( \mu \) is a Radon measure, it is inner regular at \( N \):

\[
\mu (N) = \inf \{ \mu (K) : K \subseteq N, K \text{ compact} \}.
\]

Let \( N = \bigcup A \), each \( A \) is open in \( X \) and \( \mu (A) = 0 \). Let \( K \) be compact in \( X \) and \( K \subseteq N \), then \( \exists n \in N \) s.t. \( K \subseteq \bigcup_{j=1}^{n} A_j \). Thus, \( \mu (K) \leq \sum_{j=1}^{n} \mu (A_j) = 0 \) and \( \mu (N) = 0 \).

If \( G \) is open, \( G \supseteq N \), and \( \mu (G) = 0 \) then \( G \) is one of \( A_k \) in the union \( N = \bigcup A_k \). Hence, \( G = N \).
(2) Let \( x \in \text{Supp}(\mu) \). Let \( f \in C_c(X, [0, 1]) \) be such that \( f(x) > 0 \). Let \( U = \{ y \in X : f(y) > \frac{1}{2} f(x) \} \).

Then \( U \) is open since \( f \) is continuous, and \( x \in U \).

By (1), \( \mu(U) > 0 \). Thus

\[
\int_X f \, d\mu \geq \int_U f \, d\mu = \frac{1}{2} f(x) \mu(U) > 0.
\]

Conversely, assume \( x \notin \text{Supp}(\mu) \). Let \( U = (\text{Supp}(\mu))^c \).

So, \( U \) is open, \( x \notin U \). Let \( K = \{ x \} \). Then, \( K \) is compact and \( K \subseteq U \). By Urysohn's Lemma, there exists \( f \in C_c(X, [0, 1]) \) such that \( K \leq f \leq U \).

Thus \( f(x) = 1 \) as \( x \in K \) and \( f = 1 \) on \( K \). Since \( \text{Supp}(f) \subseteq U = (\text{Supp}(\mu))^c \), we have

\[
\int_{\text{Supp}(\mu)} f \, d\mu = 0.
\]
4. Clearly, $\nu(E) > 0 \forall E \in \mathcal{B}_X$, and $\nu(\emptyset) = 0$. If $E_j \in \mathcal{B}_X$ ($j \in \mathbb{N}$) are disjoint and $E = \bigcup_{j=1}^\infty E_j \in \mathcal{B}_X$, then $\nu(E) = \int_X \phi \, d\mu = \int_X \chi_E \phi \, d\mu = \int_X \left( \sum_{j=1}^\infty \chi_{E_j} \phi \right) \, d\mu = \lim_{n \to \infty} \int_X \left( \sum_{j=1}^n \chi_{E_j} \phi \right) \, d\mu = \lim_{n \to \infty} \sum_{j=1}^n \nu(E_j) = \sum_{j=1}^\infty \nu(E_j)$, where (A) is true by the Monotone Convergence Theorem as $\phi > 0$ and $\phi \in L^1(\mu)$. Thus, $\nu$ is a Borel measure.

Now, we verify that $\nu$ is a Radon measure.

(1) Since $\phi \in L^1(\mu)$ and $\phi > 0$, $\nu$ is finite on any Borel sets, hence, compact sets.

(2) Let $E \in \mathcal{B}_X$. Let $\varepsilon > 0$. By Corollary 3.6 (which states that $\int \phi \, d\mu$ is absolutely continuous on $A$), $\exists J > 0$ s.t. $\int_A \phi \, d\mu < \frac{\varepsilon}{2}$ if $A \in \mathcal{B}_X$ and $\mu(A) < J$. 


Let \( A_k = \left\{ \frac{1}{k} \leq \phi \leq 2k \right\} \) \((k = 1, 2, \ldots)\). Then \( A_k \subseteq B_\infty \) and \( A_k \uparrow \{ \phi > 0 \} \). By the monotone convergence theorem, \( \nu(A_k \cap E) = \int \phi K_{A_k \cap E} \, dm \to \int \phi K_{\{ \phi > 0 \} \cap E} \, dm \)

\[ = \int \phi \, dm = \nu(E) \]  

Thus, \( \exists N \) s.t. \( \nu(A_N \cap E) > \nu(E) - \frac{\epsilon}{2} \).

Since \( \nu(A_N \cap E) \leq \nu(A_N) \leq \int 2N \phi \leq 2N \int \phi \, dm < \infty \), \( \nu \) is regular at \( A_N \cap E \). Thus, \( \exists K \subseteq \text{compact}, K \subseteq A_N \cap E \subseteq E \), s.t. \( \nu(K) > \nu(A_N \cap E) - \frac{\epsilon}{2} > \nu(E) - \frac{\epsilon}{2} > \nu(E) - \epsilon \). Hence, \( \nu \) is inner regular at \( E \) (any \( E \subseteq B_\infty \), not necessary open).

(3) If \( E \subseteq B_\infty \) then \( E^c \subseteq B_\infty \). By (2), \( \nu \) is inner regular at \( E^c \). Thus, \( \forall \epsilon > 0, \exists \text{ compact } F \subseteq E^c \) s.t. \( \nu(F) > \nu(E^c) - \epsilon \). Let \( U = F^c \). Since \( F \) is compact, it is closed. So, \( U = F^c \) is open. Since \( F \subseteq E^c \), \( U = F^c \supseteq E \).

Moreover, \( \nu(U) = \nu(F^c) = \nu(X) - \nu(F) < \nu(X) - \nu(E^c) + \epsilon = \nu(E) + \epsilon \). Thus, \( \nu \) is outer regular at \( E \).

Hence (1) - (3) \( \Rightarrow \nu \) is a Radon measure.
We show now \( \text{Supp}(\phi) \subseteq \text{Supp}(\Psi) \cap \text{Supp}(\mu) \).

Equivalently, \((\text{Supp}(\Psi))_c \supseteq (\text{Supp}(\phi))_c \cup (\text{Supp}(\mu))_c \).

Let \( U = (\text{Supp}(\phi))_c \). Then \( U \) is open, and \( \phi = 0 \) on \( U \).

So, \( \nu(U) = \int \phi \, du = 0 \). Thus, \( U \subseteq (\text{Supp}(\Psi))_c \).

Similarly, let \( V = (\text{Supp}(\mu))_c \). Then \( V \) is open, and \( \mu(V) = 0 \). Thus \( \nu(V) = \int \phi \, du = 0 \).

Hence \( V \subseteq (\text{Supp}(\phi))_c \).

5. We show that \( \delta_{x_0} \) is a Radon measure. Clearly, it is a Borel measure. It is a finite measure, so it is finite on compact sets. Let \( E \subseteq \mathbb{R}^d \). If \( x \notin E \) then \( \delta_{x_0}(E) = 1 \). If \( U \) is open and \( E \subseteq U \) then \( \delta_{x_0}(U) = 1 \). Hence, \( \delta_{x_0} \) is outer regular. If \( x \notin E \), then \( \delta_{x_0}(E) = 0 \). For the open set \( U = X \setminus \{x_0\} \),
we have \( E \subseteq U \) and \( \mathcal{S}(U) = \emptyset \). Thus, \( \mathcal{S} \) is also outer regular at \( E \). Finally, let \( U \) be open if \( x \in U \). Then \( \mathcal{S}(U) = 1 \). Also, \( K = \{x_0\} \) is compact and \( K \subseteq U \), \( \mathcal{S}(K) = 1 \). Thus, \( \mathcal{S} \) is inner regular at \( U \).

If \( x \notin U \), \( \mathcal{S}(U) = 0 \). If \( K \subseteq U \), \( K \) is compact, then \( x \notin K \), so \( \mathcal{S}(K) = 0 \). Hence, \( \mathcal{S} \) is inner regular at \( U \).

Thus, \( \mathcal{S} \) is inner regular at open sets, and therefore \( \mathcal{S} \) is a Radon measure.

\[
\forall f \in C_c(X) \; \int f \, d\mathcal{S} = \int f(x_0) \, d\mathcal{S}_{\{x_0\}} = f(x_0) \, \mathcal{S}(\{x_0\}) = f(x_0) = I(f).
\]

By the uniqueness in the Riesz Representation Theorem, \( \mathcal{S} \) is the Radon measure associated with the functional \( I \).
6. Let $K = \text{supp}(\mu)$. $K$ is a closed subset of $X$. Since $X$ is compact, $K$ is compact. Moreover, $\mu(K^c) = 0$. Hence $\mu(K) = \mu(K) + \mu(K^c) = \mu(X) = 1$.

Let $H$ be a compact subset of $X$. Assume $H \not\subseteq K$ (i.e., $H \subseteq K$ but $H \neq K$). Then $\exists x \in K \setminus H$. In particular, $\exists$ open sets $U$ and $V$ s.t. $x \in U$, $H \subseteq V$, and $U \cap V = \emptyset$. (See Proposition 4.23.) Since $x \in K = \text{supp}(\mu)$, $\mu(U) > 0$. [Otherwise $U \subseteq \text{supp}(\mu)^c$ and $x \notin U$.] Since $\mu(K) = 1$, $\mu(U \cup K) = \mu(U) > 0$. [$1 = \mu(X) \geq \mu(U \cup K) \geq \mu(K) = 1 \implies \mu(U \cup K) = 1$. But $\mu(U) + \mu(K) = \mu(U \cup K) + \mu(U \cup K)$. So, $\mu(U \cup K) = \mu(U) > 0$.] Since $H \cup (U \cup K) \subseteq K$ and $H \cap (U \cup K) = \emptyset$, we have $\mu(H) + \mu(U \cup K) \leq \mu(K) = 1$. $\mu(H) \leq 1 - \mu(U \cup K) < 1$. 
7. Let \( \mu \) be a Borel measure on \( \mathbb{R}^n \), which is an LCH space. Assume \( \mu \) is a Radon measure. Then by definition \( \mu \) is finite on compact sets. Conversely, assume that \( \mu \) is finite on all compact subsets of \( \mathbb{R}^n \). If \( U \neq \emptyset \) is an open subset of \( \mathbb{R}^n \), then it is the countable union of compact sets \( \overline{B(x, \varepsilon)} \), where \( x \in U \), \( \varepsilon \in (0, \infty) \cap \mathbb{Q} \), and \( B(x, \varepsilon) = \{ y \in \mathbb{R}^n : |y - x| < \varepsilon \} \) such that \( \overline{B(x, 2\varepsilon)} \subset U \). Thus, by Theorem 7.8, \( \mu \) is a Radon measure.