1. If $f \in C_c(X)$ and $f \geq 0$ on $X$, then $\int_Y f \, d\mu \geq 0$. Thus $I$ is positive. Since the restriction $f|_Y = X Y f$ and the integration are both linear, $I$ is linear. Verify that $\nu$ is a Radon measure on $X$.

(1) Let $K \subseteq X$ be compact. So, $K$ is closed, and $K \cap Y$ is closed. So, $K \cap Y$ is compact in $X$. Let $\{U_n \cap Y\}_{n=1}^\infty$ be an open cover of $K \cap Y$ in $Y$, where $U_n$ is open in $X$. Then $U_n \cap (K \cap Y)$, $Y$, form an open cover of $K \cap Y$ in $X$. So, $\exists \{V_1, \ldots, V_m\} \subseteq Y$ such that $\bigcup_{i=1}^m V_i$ is a cover of $K \cap Y$ in $X$. Hence, $\{U_n \cap Y\}_{n=1}^\infty$ covers $K \cap Y$, and $K \cap Y$ is compact in $Y$. Thus $\nu(K) = \mu(K \cap Y) < \infty$. 
(2) If $E$ is a Borel subset of $X$ then $E \cap Y$ is a Borel subset of $Y$. Since $\mu$ is a Radon measure on $Y$, it is outer regular at $E \cap Y$, i.e.,

$$\mu(E \cap Y) = \inf \{ \mu(U \cap Y) : U \text{ open in } X, U \cap Y \supseteq E \cap Y \}$$

(Open sets in $Y$ are of the form $U \cap Y$, $U$ open in $X$.)

Thus, by the def. of $\nu$,

$$\nu(E) = \inf \{ \nu(U) : U \text{ open in } X, U \cap Y \supseteq E \cap Y \} =: a.$$ 

Let $b = \inf \{ \nu(U) : U \text{ open in } X, U \supseteq E \cap Y \}$. We have $a \leq b$ since $U \supseteq E \Rightarrow U \cap Y \supseteq E \cap Y$. If $U$ is open in $X$ and $U \cap Y \supseteq E \cap Y$ then for $V = U \cap Y$, open in $X$, $V \supseteq E$, and $\nu(V) = \mu(V \cap Y) = \mu(U \cap Y) = \mu(U)$.

Thus $a \leq b$. Hence $a = b$, and $\nu$ is outer regular.

(3) Let $U$ be open in $X$. Then $U \cap Y$ is open in $Y$. Since $\mu$ is inner regular at $U \cap Y$, 

...
\[ \nu(U) = \mu(U \cap Y) \]
\[ = \inf \{ \mu(F) : F \text{ compact in } Y, F \subseteq U \} \]
\[ = \inf \{ \mu(F \cap Y) : F \text{ compact in } Y, F \subseteq U \} = c. \]

Denote \( d = \inf \{ \nu(K) : K \text{ compact in } X, K \subseteq U \} \)
\[ = \inf \{ \mu(K \cap Y) : K \text{ compact in } X, K \subseteq U \}. \]

If \( F \) is compact in \( Y \) and \( F \subseteq U \), then \( F \) is also compact in \( X \) as \( Y \) is closed in \( X \). Thus \( d \leq c \).

Let \( K \) be compact in \( X \) and \( K \subseteq U \). Let \( F = K \cap Y \).

Then, \( F \) is compact in \( Y \) (cf. Part 1) and \( F \subseteq U \).
Moreover, \( \mu(K \cap Y) = \mu(F \cap Y) \geq c. \) Hence, \( d \geq c. \)

Thus, \( d = c \), and \( \nu \) is inner regular at \( U \).

By (1) – (3), \( \nu \) is a Radon measure on \( X \). Namely, \( \forall f \in C_c(X), \]
\[ \int f \, d\nu = \int f \, d\nu|_Y + \int f \, d\nu = \int f \, d\mu|_Y = I(f), \]
\[ \int f \, d\nu|_Y = \int f \, d\nu|_Y \]

By the uniqueness, \( \nu \) is exactly the Radon measure associated with \( I \).
2. Let $K \subseteq X$ be compact. Let $\phi_K \in C_c(X, [0, 1])$ be such that $K \subseteq \phi_K \subseteq X$. Let $f \in C_c(X)$ with $\text{supp}(f) \subseteq K$. Then $\|f\|_1 \phi_K \pm f \geq 0$ on $X$. So, $I(\|f\|_1 \phi_K \pm f) \geq 0$, i.e., $\|f\|_1 I(\phi_K) \pm I(f) \geq 0$. Hence, $|I(f)| \leq I(\phi) \|f\|_1$. Set $C_K = I(\phi_K)$.

3. (1) Clearly $N$ is open. Since $\mu$ is a Radon measure, it is inner regular at $N$:

$$\mu(N) = \inf \{ \mu(K) : K \subseteq N, \ K \text{ compact} \}.$$

Let $N = \bigcup A_k$, each $A_k$ is open in $X$ and $\mu(A_k) = 0$. Let $K$ be compact in $X$ and $K \subseteq N$, then $\exists n \in N$ s.t. $K \subseteq \bigcup_{j=1}^{\infty} A_{k_j}$. Thus, $\mu(K) \leq \sum_{j=1}^{\infty} \mu(A_{k_j}) = 0$ and $\mu(N) = 0$. If $G$ is open, $G \supseteq N$, and $\mu(G) = 0$ then $G$ is one of $A_k$ in the union $N = \bigcup A_k$. Hence, $G = N$. 

(2) Let \( x \in \text{Supp}(u) \). Let \( f \in \mathcal{C}_c(X, [0,1]) \) be such that \( f(x) > 0 \). Let \( U = \{ y \in X : f(y) > \frac{1}{2} f(x) \} \).

Then \( U \) is open, since \( f \) is continuous, and \( x \in U \).

By (1), \( m(U) > 0 \). Thus

\[
\int_X f \, du \geq \int_U f \, du \geq \frac{1}{2} f(x) \cdot m(U) > 0.
\]

Conversely, assume \( x \notin \text{Supp}(u) \). Let \( U = (\text{Supp}(u))^c \).

So, \( U \) is open, \( x \in U \). Let \( K = \{ x \} \). Then, \( K \) is compact and \( K \subseteq U \). By Urysohn's Lemma, there exists \( f \in \mathcal{C}_c(X, [0,1]) \) such that \( K \subsetneq f \subseteq U \).

Thus \( f(x) = 1 \) as \( x \in K \) and \( f = 1 \) on \( K \). Since \( \text{Supp}(f) \subseteq U = (\text{Supp}(u))^c \), we have

\[
\int_{\text{Supp}(u)} f \, du = 0.
\]
4. Clearly \( \nu(E) > 0 \) \( \forall E \in \mathcal{B}_X \), and \( \nu(\emptyset) = 0 \).

If \( E_j \subseteq \mathcal{B}_X \) \( (j \in \mathbb{N}) \) are disjoint and \( E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{B}_X \), then
\[
\nu(E) = \int_E \phi \, d\mu = \int_X (\sum_{j=1}^{\infty} \chi_{E_j} \phi) \, d\mu
\]
\[
= \lim_{n \to \infty} \int_X \left( \sum_{j=1}^{n} \chi_{E_j} \phi \right) \, d\mu = \lim_{n \to \infty} \sum_{j=1}^{n} \nu(\mathcal{E}_j) = \sum_{j=1}^{\infty} \nu(\mathcal{E}_j)
\]
where \((A)\) is true by the Monotone Convergence Theorem.

as \( \phi \geq 0 \) and \( \phi \in L^1(\mu) \). Thus \( \nu \) is a Borel measure.

Now, we verify that \( \nu \) is a Radon measure.

(1) Since \( \phi \in L^1(\mu) \) and \( \phi \geq 0 \), \( \nu \) is finite on any Borel sets, hence, compact sets.

(2) Let \( E \in \mathcal{B}_X \). Let \( \varepsilon > 0 \). By Corollary 3.6 (which states that \( \int \phi \, d\mu \) is absolutely continuous on \( A \)), \( \exists \delta > 0 \) s.t. \( \int_A \phi \, d\mu < \varepsilon \) if \( A \subseteq \mathcal{B}_X \) and \( \mu(A) < \delta \).
Since $\mathcal{M}$ is Radon, it is outer regular at $E$. Hence $\exists U$ open, $U \supseteq E$ s.t. $m(U \setminus E) = m(U) - m(E) < \varepsilon$. Thus,

\[ \nu(U) - \nu(E) = \nu(U \setminus E) = \int_{U \setminus E} \phi \, dm < \varepsilon. \]

Thus, $\nu(E) = \inf \{ \nu(U) : U \text{ open, } U \supseteq E \}$ i.e., $\nu$ is outer regular at $E$, and $\nu$ is outer regular.

(3) Let $U$ be open. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $A \subset B_x$ and $m(A) < \delta \implies \int_A \phi \, dm < \varepsilon$. Now, $\mathcal{M}$ is Radon, so it is inner regular at $U$. So, there exists compact set $K \subseteq U$ s.t. $m(U \setminus K) < \delta$. Hence $\nu(U \setminus K) = \int_{U \setminus K} \phi \, dm < \varepsilon$. Thus, $\nu(K) \leq \nu(U) < \nu(K) + \varepsilon$.

So, $\nu(U) = \sup \{ \nu(K) : K \text{ compact, } K \subseteq U \}$ i.e., $\nu$ is inner regular at $U$. So, it is inner regular at opens. (1) $\rightarrow$ (3) $\implies \nu$ is Radon.
We show now \( \text{supp}(v) \subseteq \text{supp}(\phi) \lor \text{supp}(\mu) \).

Equivalently, \( (\text{supp}(v))^c \supseteq (\text{supp}(\phi))^c \lor (\text{supp}(\mu))^c \).

Let \( U = (\text{supp}(\phi))^c \). Then \( U \) is open, and \( \phi = 0 \) on \( U \).

So, \( \nu(U) = \int_U \phi \, du = 0 \). Thus, \( U \subseteq (\text{supp}(v))^c \).

Similarly, let \( V = (\text{supp}(\mu))^c \). Then \( V \) is open, and \( \mu(V) = 0 \). Thus, \( \nu(V) = \int_V \phi \, du = 0 \).

Hence \( V \subseteq (\text{supp}(\phi))^c \).

5. We show that \( \nu_0 \) is a Radon measure. Clearly it is a Borel measure. It is a finite measure, so it is finite on compact sets. Let \( E \subseteq \mathbb{R}^n \). If \( x \notin E \) then \( \nu_0(E) = 0 \). If \( U \) is open and \( E \subseteq U \) then \( \nu_0(U) = 1 \). Hence, \( \nu_0 \) is outer regular at \( E \).

If \( x \notin E \), then \( \nu_0(E) = 0 \). For the open set \( U = X \setminus \{x_0,3\} \),
we have \( E \subseteq U \) and \( \Delta_{x_0}(U) = 0 \). Thus, \( \Delta_{x_0} \) is also outer regular at \( E \). Finally, let \( U \) be open. If \( x_0 \in U \) then \( \Delta_{x_0}(U) = 1 \). Also, \( K = \{ x_0 \} \) is compact and \( K \subseteq U \), \( \Delta_{x_0}(K) = 1 \). Thus, \( \Delta_{x_0} \) is inner regular at \( U \). If \( x_0 \notin U \), \( \Delta_{x_0}(U) = 0 \). If \( K \subseteq U \), \( K \) is compact, then \( x_0 \notin K \). So \( \Delta_{x_0}(K) = 0 \). Hence, \( \Delta_{x_0} \) is inner regular at \( U \).

Thus, \( \Delta_{x_0} \) is inner regular at open sets and therefore \( \Delta_{x_0} \) is a Radon measure.

\( \forall f \in C_c(X) \) \( \int f \, d\Delta_{x_0} = \sum f(x_0) \Delta_{x_0}(\{x_0\}) = f(x_0) \cdot \delta_{x_0} \{x_0\} = I(f) \).

By the uniqueness in the Riesz representation, \( \Delta_{x_0} \) is the Radon measure associated with the functional \( I \).
6. Let \( K = \text{supp}(\mu) \). \( K \) is a closed subset of \( X \). Since \( X \) is compact, \( K \) is compact. Moreover, \( m(K^c) = 0 \). Hence \( m(K) = m(K) + m(K^c) = m(X) = 1 \).
   Let \( H \) be a compact subset of \( X \). Assume \( H \not\subseteq K \) (i.e., \( H \subseteq K \) but \( H \not= K \)). Then \( \exists x \in K \setminus H \). In particular, \( \exists \) open sets \( U \) and \( V \) s.t. \( x \in U \), \( H \subseteq V \), and \( U \cap V = \emptyset \). (See Proposition 4.23.)
   Since \( x \in K = \text{supp}(\mu) \), \( \mu(U) > 0 \). [Otherwise \( U \subseteq (\text{supp}(\mu))^c \) and \( x \not\in U \).] Since \( m(K) = 1 \), \( m(U \cap K) = m(U) > 0 \). \([1 = m(X) \geq m(U \cup K) \geq m(K) = 1 \Rightarrow m(U \cup K) = 1 \]. But \( m(U) + m(K) = m(U \cap K) + m(U \cup K) \). So, \( m(U \cap K) = \mu(U) > 0 \).]
   Since \( H \subseteq (U \cup K) \) and \( H \cap (U \cap K) = \emptyset \), we have \( m(H) + m(U \cap K) \leq m(K) = 1 \). \( m(H) \leq 1 - m(U \cap K) < 1 \).
7. Let $\mu$ be a Borel measure on $\mathbb{R}^n$, which is an LCHI space. Assume $\mu$ is a Radon measure. Then by definition $\mu$ is finite on compact sets.

Conversely, assume that $\mu$ is finite on all compact subsets of $\mathbb{R}^n$. If $U \neq \emptyset$ is an open subset of $\mathbb{R}^n$, then it is the countable union of compact sets $\overline{B(x, \varepsilon)}$, where $x \in U \cap \mathbb{Q}^n$, $\varepsilon \in (0, \infty) \cap \mathbb{Q}$, and $B(x, \varepsilon) = \{y \in \mathbb{R}^n : |y-x| < \varepsilon \}$ such that $B(x, 2\varepsilon) \subset U$. Thus, by Theorem 7.8, $\mu$ is a Radon measure.