Math 240C, Spring 2020
Solution to Problems of HW #2
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1. (1) Let $U$ be open. Let $f = \phi Xu$. Let $x \in \mathbb{R}$.
We show $\{\phi Xu > x\}$ is open. Thus, $\phi Xu$ is LSC.
In fact, if $x < 0$ then $\{\phi Xu > x\} = X$, if $x = 0$ then $\{\phi Xu > 0\} = U$; if $x > 0$ then $\{\phi Xu > x\} = U \cap \{\phi > x\}$ is open. Now, by Corollary 7.13, applied to $\phi Xu$,
$$
\nu(U) = \int \phi Xu \, dx = \sup \left\{ \int g \, dx : g \in C_c(X), 0 \leq g \leq \phi Xu \right\},
$$
Since $\nu'$ is the Radon measure associated to $f \mapsto \int f \phi \, dx$, then
$$
\nu'(U) = \sup \left\{ \int f \phi \, dx : f \in C_c(X, [0,1]), f < U \right\}.
$$
If $f \in C_c(X, [0,1])$ and $f < U$, then $g := f\phi \in C_c(X)$ and $0 \leq g = f\phi \leq \phi Xu$. Thus, $\nu'(U) \leq \nu(U)$. 
We now show that $V'(U) \leq V(U)$. Assume that $V'(U) < \infty$ for otherwise $V'(U) \geq V(U)$.

Let $f \in C_c(X)$ be such that $0 \leq f \leq \chi_U$. Denote $U_f = \{ f > 0 \}$. Clearly, $U_f \subseteq U$. Moreover, $U_f$ is open as $f$ is continuous, and $\overline{U_f} = \text{supp}(f)$ is compact. Let $\varepsilon > 0$. Since $\mu$ is a Radon measure on $X$, it is inner regular at the open set $U_f$.

Thus there exists a compact set $K \subseteq U_f$ s.t. $\mu(U_f \setminus K) = \mu(U_f) - \mu(K) < \varepsilon/(2 \| f \|_{U_f})$, where

$$\| f \|_{U_f} = \max_{x \in U_f} |\phi(x)| > 0.$$ (Note that $\mu(U_f) \leq \mu(\overline{U_f}) < \infty$.)

Now, by Urysohn’s lemma, $\exists f_K \in C_c(X, [0, 1])$ s.t. $K < f_K < U_f$. Then, $f_K < U$ and hence

$$\int_X f_K \phi \, d\mu \leq V'(U).$$

In the meantime, we have

$$\int_X f \phi \, d\mu = \int_{U_f} f \phi \, d\mu = \int_{U_f} f_K \phi \, d\mu + \int_{U_f \setminus K} f \phi \, d\mu = f_K \int_{U_f} \phi \, d\mu + \int_{U_f \setminus K} f \phi \, d\mu,$$

which concludes the proof.
\[ \leq \int_k \phi \, du + \| \phi \|_{L^1} \mu(U_k \setminus K) + \int_k \phi \, du + z/2. \]

Also, \[ \int_k \phi \, du = \int_k \phi \, du \leq \int_k \chi \phi \, du \leq \int K \phi \, du. \]

Thus, \[ \int \phi \, du - \int_k \phi \, du \leq \int K \phi \, du - \int K \phi \, du + z/2 \]
\[ = \int \phi \, du + z/2 \leq \| \phi \|_{L^1} \mu(U_k \setminus K) + z/2 < z. \]

This implies that \[ \int \phi \, du < z + \int \phi \, du \leq z + V'(U). \]

Hence \[ \int \phi \, du \leq V'(U). \] Thus, \[ V(U) \leq V'(U). \]

(2) Let \( E \subset \mathbb{R}^1 \). We may assume that \( V(E) < \infty \) for otherwise \( V \) is outer regular at \( E \). Let \( V_k = \{ 2^k < \phi < 2^{k+1} \} \) \((k \in \mathbb{Z})\). Then each \( V_k \) is open and \( \bigcup_k V_k = \mathbb{R}^1 \). Hence \( E = \bigcup_k (E \cap V_k) \). Each \( \mu(E \cap V_k) < \infty \), since \( V(E) < \infty \). Moreover, for each \( k \in \mathbb{Z} \), we have
\[ \forall \epsilon > 0 \Rightarrow \nu(E \cap V_k) = \int_{E \cap V_k} \phi \, dm \leq 2^{k+2} \int_{E \cap V_k} dm = 2^{k+2} \mu(E \cap V_k). \]

Hence \( \mu(E \cap V_k) < \infty \) \( \forall k \in \mathbb{Z} \). Now let \( \epsilon > 0 \). Since \( \mu \) is a Radon measure, it is outer regular at each \( E \cap V_k \). Thus, there exist open sets \( U_k \supseteq E \cap V_k \) s.t. \( \mu(U_k \setminus (E \cap V_k)) < \epsilon / (3 \cdot 2^{1k}) \). Let \( U_k = U_k \cap V_k \).

Then each \( U_k \) is open, \( U_k \supseteq E \cap V_k \), and
\[ \mu(U_k \setminus (E \cap V_k)) \leq \mu(U_k \setminus (E \cap V_k)) < \epsilon / (3 \cdot 2^{1k}) \]
Let \( U = \bigcup_k U_k \).

Then \( U \) is open, \( U \supseteq \bigcup_k (E \cap V_k) = E \setminus (\bigcup_k V_k) = E \). Hence
\[ U = (\bigcup_k U_k) \setminus \bigcup_j (E \setminus V_j), \quad U \in E \subseteq \bigcup_k (U_k \setminus (E \cap V_k)) \cup E. \]

\[ \nu(U) \leq \sum_k \nu(U_k \setminus (E \cap V_k)) + \nu(E) \leq \sum_k \int_{E \cap V_k} \phi \, dm + \nu(E) \]
\[ \leq \sum_k 2^{k+2} \mu(U_k \setminus (E \cap V_k)) + \nu(E) < \epsilon + \nu(E). \]

Thus, \( \nu \) is outer regular.

(3) Both \( \nu' \), \( \nu \) are outer regular, agree on open sets. So, \( \nu' = \nu \). But \( \nu' \) is Radon, so is \( \nu \).
2. Since \( \mu \) is a Radon measure and \( \mu(A) < \infty \), 
\( \mu \) is inner regular at \( A \). Thus there exists a compact subset \( K \) of \( X \) such that \( K \subseteq A \) and \( \mu(K) > \delta \).

We construct a sequence of compact subsets \( K_j \) (\( j = 1, 2, \ldots \)) of \( X \) such that

- \( K \supseteq K_1 \supseteq \cdots \supseteq K_j \supseteq \cdots \)
- \( \forall j \geq 1: \alpha \leq \mu(K_j) \leq \alpha + \frac{\delta}{2^j} \) (\( j = 1, 2, \ldots \)).

If for some \( j^* \), \( \alpha = \mu(K_{j^*}) \), then we choose all \( K_{j+n} = K_j \) (\( n = 1, 2, \ldots \)). So, we may assume \( \alpha < \mu(K_j) \leq \alpha + \frac{\delta}{2^j} \). Once all these \( K_j \)'s are constructed, then \( \beta = \bigcap_{j=1}^{\infty} K_j \subseteq K \subseteq A \) and 
\( \mu(\beta) = \lim_{j \to \infty} \mu(K_j) = \alpha \), as desired.
\( V_x \subset K. \) Since \( m(\{x\}) = 0 \) and \( K \) is outer regular, there exists an open set \( U_x \supset x \) such that \( m(U_x) < \frac{\alpha}{2}. \) Since \( \{x\} \) is compact, \( \{x\} \subset \overline{U_x} \) and \( K \) is locally compact and Hausdorff, there exists a precompact open set \( V_x \) such that \( x \in V_x \subset \overline{V_x} \subset \overline{U_x} \) (cf. Prop. 7.31). Note that \( m(\overline{V_x}) = m(\overline{U_x}) < \frac{\alpha}{2}. \) Now \( \{V_x : x \in K\} \) covers \( K, \) which is compact. Thus, there is a finite subcover \( V_{x_1} \ldots V_{x_m}. \) Let \( F_j = (\bigcup_{i=1}^{j} V_i) \cap K, \ j = 1, \ldots, m. \) Then each \( F_j \) is compact, \( F_1 \subset \ldots \subset F_m = K, \) \( m(F_1) \leq m(V_1) < \frac{\alpha}{2}, \ m(F_m) = m(K) > \alpha. \) \( m(F_{j+1}) - m(F_j) \leq m(V_{j+1}) < \frac{\alpha}{2} \) \( (j = 1, \ldots, m-1). \) Thus, there exists \( j \) such that \( m(F_{j+1}) \leq \alpha \leq m(F_j). \) Let \( K_1 = F_j. \) Then \( K_1 \) is compact, \( K_1 \subset K, \) \( 0 \leq m(K_1) - \alpha \leq m(F_j) - m(F_{j-1}) < \frac{\alpha}{2}. \) Replace \( K \) by \( K_1, \) \( \alpha/2 \) by \( \alpha/2^2, \) etc., we can obtain \( K_2, \) \( \ldots \) An induction provides all \( K_j (j = 1, 2, \ldots) \) as needed.
3. Write \( f = f^+ - f^- \) with \( f^\pm = \max(\pm f, 0) \). Then \( f^+ \geq 0 \) and \( f^\pm \in L^1(\mu) \). Let \( E_k = \{ \mu : |f| < k \} \) (\( \forall k \in \mathbb{N} \)). Then \( \mu(E_k) \leq k \int |f| \, d\mu < \infty \). Hence \( \{ |f| > 0 \} = \bigcap_{k=1}^\infty E_k \) is \( \sigma \)-finite.

It now follows from Proposition 7.14 that \( \forall \varepsilon > 0 \), there exist LSC functions \( g^+, g^- \) on \( X \) and USC functions \( h^+, h^- \) s.t. \( 0 \leq f^\pm \leq g^\pm \), \( 0 \leq h^\pm \leq f^\pm \), and
\[
\int g^\pm \, d\mu < \int f^\pm \, d\mu + \varepsilon/4 \quad \text{and} \quad \int f^\pm \, d\mu < \int h^\pm \, d\mu + \varepsilon/4.
\]

Let \( g = g^+ - h^- \) and \( h = h^+ - g^- \). Then, \( g \) is an LSC and \( h \) is an USC function. Moreover, \( h = h^+ - g^- \leq f^+ - f^- = f \leq g^+ - h^- = g \), and
\[
\int (g - h) \, d\mu = \int g^+ \, d\mu - \int h^- \, d\mu - \int h^+ \, d\mu + \int g^- \, d\mu < \int f^+ \, d\mu + \varepsilon/4 - \int f^- \, d\mu + \varepsilon/4
\]
\[
-\int f^+ \, d\mu + \varepsilon/4 + \int f^- \, d\mu + \varepsilon/4
\]
\[= \varepsilon. \quad \Box \]
4. If $C([0,1]) \subset C([0,1])^{**}$, then $C([0,1])^{**}$ would be separable. Thus, $C([0,1])^*$ would be separable (cf. Exercise 25 on page 165 of the textbook or Problem 5 of HW #4, Math 240 B, Winter 2020).

For $x \in [0,1]$, define $F_x \in C([0,1])^*$ by $F_x(f) = f(x)$ for all $f \in C([0,1])$. Note that $|F_x(f)| \leq ||f|| \Rightarrow ||F_x|| \leq 1$. If $f \equiv 1$ on $[0,1]$ then $||f|| = 1$ and $|F_x(f)| = 1$. Hence $||F_x|| = 1$. If $x_1, x_2 \in [0,1]$ and $x_1 \neq x_2$, then $||F_{x_1} - F_{x_2}|| = 1$, since for $g \in C([0,1])$,

$$g(x_1) = 1, \ g(x_2) = 0, \ ||g|| = 1, \ \ |(F_{x_1} - F_{x_2})(g)| = 1.$$ 

Thus, $C([0,1])^*$ has uncountably many balls $B(F_x, \frac{1}{2})$ ($x \in [0,1]$, that are pairwise disjoint. Hence, $C([0,1])^*$ is not separable, and $C([0,1])^{**} \neq C([0,1])$. QED
5. If \( f_n \to f \) weakly then \( \sup_{n \geq 1} \|f_n\| < \infty \). [This is true for any Banach space \( B \). \( u_n \to u \) weakly in \( B \) means that \( g(u_n) \to g(u) \) \( \forall g \in B^* \). But each \( b \in B \) defines \( b^{**} \in B^{**} \); \( b^{**}(f) = f(b) \) \( \forall f \in B^* \) with \( \|b^{**}\| = \|b\| \). Thus, \( u_n \to u \) weakly means \( u_n^{**}(f) \to u^{**}(f) \) \( \forall f \in B^* \). Thus, the Principle of Uniform Boundedness implies that \( \sup_{n \geq 1} \|f_n\| < \infty \).]

\( \forall x \in X \). Def \( f_x(f) = f(x) \) then \( f_x \) is linear on \( C_0(X) \).

\[ |f_x(f)| = |f(x)| \leq \|f\| \quad \forall f \in C_0(X). \] So \( f_x \in C_0(X)^* \) \( \forall f \in C_0(X) \).

Let \( K = \{x \} \) compact. By Urysohn's lemma, \( \exists f_x \in C(X, [0, 1]) \) s.t. \( K < f_x < X \). So, \( f_x(f_x) = f_x(x) = 1 \), \( \|f_x\| \leq 1 \). Thus, \( \|f_x\| = 1 \).

Now, \( f_n \to f \) \( \Rightarrow f_n(f_n) \to f(f) \), i.e. \( f_n(x) \to f(x) \) \( \forall x \in X \).

Now, let \( F \in C_0(X)^* \). Then \( \exists M \in M(X) \) s.t. \( F(f) = \int f \, d\mu \) \( \forall f \in C_0(X) \).

Since \( f_n \to f \) pointwise on \( X \), \( \sup_{n \geq 1} \|f_n\| < \infty \) and \( M(X) < \infty \), the dominated convergence theorem \( \Rightarrow F(f_n) = \int f_n \, d\mu \to \int f \, d\mu = F(f) \).

Thus, \( f_n \to f \) weakly. \( \Box \)
6. (1) Since any \( f \in C([0,1]) \) is Riemann integrable, we have \( \langle m_n, f \rangle = \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) \rightarrow \int_0^1 f(x) \, dx = \int f \, dm_{[0,1]} \) = \( \langle m, f \rangle \), i.e. \( m_n \rightarrow m \) vaguely.

(2) Let \( E = \mathbb{Q} \cap [0,1] = \{ \text{all rational numbers in } [0,1] \} \). Then \( m_n(E) = 1 \) for all \( n \geq 1 \). But \( m(E) = 0 \). Hence \( m_n(E) \not\rightarrow m(E) \).