1. Let $X$ be a TVS, $x_0 \in X$, and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$. Prove that the maps $x \mapsto x + x_0$ and $x \mapsto \alpha x$ are homeomorphisms of $X$.

2. Let $X$ be a TVS and $U$ the collection of all open sets of $X$ that contain 0. Prove the following:
   (1) For any $A \subseteq X$, $\bar{A} = \bigcap_{U \in U} (A + U)$;
   (2) For any $U \in U$, there exists a balanced open subset $V$ of $X$ such that $0 \in V \subseteq U$;
   (3) If $U \in U$ and $\{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha_1 < \cdots < \alpha_n < \cdots$ and $\alpha_n \to \infty$, then $X = \bigcup_{n=1}^{\infty} \alpha_n U$;
   (4) If $U \in U$ is bounded and $\{\epsilon_n\}$ is a sequence of real numbers such that $\epsilon_1 > \cdots > \epsilon_n > \cdots$ and $\epsilon_n \to 0$. Then $\{\epsilon_n U\}_{n=1}^{\infty}$ is a local base at 0.

3. Prove that any compact subset of a TVS is bounded.

4. Let $X$ be a Hausdorff TVS. Prove the following:
   (1) Let $F$ be a closed subset of $X$ and $x \in X \setminus F$. Then there exist open sets $U$ and $V$ of $X$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$;
   (2) Let $K$ be a compact subset of $X$ and $F$ a closed subset of $X$ with $K \cap F = \emptyset$. Then there exist open sets $U$ and $V$ of $X$ such that $K \subseteq U$, $F \subseteq V$, and $U \cap V = \emptyset$.

5. Let $X$ be an LCS defined by a family of seminorms $\mathbb{P}$. Let $A \subseteq X$. Prove that the following are equivalent:
   (1) The subset $A$ is bounded;
   (2) For any continuous seminorm $q$ on $X$, $\sup \{q(a) : a \in A\} < \infty$;
   (3) For any $p \in \mathbb{P}$, $\sup \{p(a) : a \in A\} < \infty$.

6. Prove the following:
   (1) Assume that $X$ is a finitely dimensional vector space and that $(X, \tau_1)$ and $(X, \tau_2)$ are two Hausdorff TVSs, then $\tau_1 = \tau_2$;
   (2) A Hausdorff TVS is finitely dimensional if and only if it is locally compact.

7. Prove the following:
   (1) Any vector subspace of a TVS is also a TVS with respect to the subspace topology;
   (2) The closure of any subspace of a TVS is also a subspace;
   (3) Any finitely dimensional subspace of a Hausdorff TVS is closed.

8. Let $X$ and $Y$ be two LCS. Let $Q$ be a family of seminorms that define the topology of $Y$. Let $T : X \to Y$ be a linear map. Prove that the following are equivalent:
   (1) $T : X \to Y$ is continuous;
   (2) For any continuous seminorm $p$ on $Y$, $p \circ T$ is a continuous seminorm on $X$;
   (3) For any $q \in Q$, $q \circ T$ is a continuous seminorm on $X$.

9. Let $X$ be an LCS topologized by a sequence of seminorms $\{p_n\}_{n=1}^{\infty}$ on $X$ such that $\cap_{n=1}^{\infty} \{x \in X : p_n(x) = 0\} = \{0\}$. Let
   \[
   d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)} \quad \forall x, y \in X.
   \]
   Prove that $(X, d)$ is a metric space and its topology is the same as that induced by the seminorms $\{p_n\}_{n=1}^{\infty}$.

10. Let $X$ be an LCS. Prove that $X$ is metrizable if and only if $X$ is first countable.