Math 241B: Functional Analysis
Winter, 2007

Homework Assignment 1
Due Wednesday, January 24

1. Let X be a TVS, $x_0 \in X$, and $\alpha \in \mathbb{F}$ with $\alpha \neq 0$. Prove that the maps $x \longmapsto x + x_0$ and $x \longmapsto \alpha x$ are homeomorphisms of X.
2. Let X be a TVS and $\mathbb{U}$ the collection of all open sets of X that contain 0. Prove the following:
   (1) For any $A \subseteq X$, $\bar{A} = \cap_{U \in \mathbb{U}} (A + U)$;
   (2) For any $U \in \mathbb{U}$, there exists a balanced open subset V of X such that $0 \in V \subseteq U$;
   (3) If $U \in \mathbb{U}$ and $\{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha_1 < \cdots < \alpha_n < \cdots$ and $\alpha_n \to \infty$, then $X = \cup_{n=1}^{\infty} \alpha_n U$;
   (4) If $U \in \mathbb{U}$ is bounded and $\{\epsilon_n\}$ is a sequence of real numbers such that $\epsilon_1 > \cdots > \epsilon_n > \cdots$ and $\epsilon_n \to 0$. then $\{\epsilon_n U\}_{n=1}^{\infty}$ is a local base at 0.
3. Prove that any compact subset of a TVS is bounded.
4. Let X be a Hausdorff TVS. Prove the following:
   (1) Let F be a closed subset of X and $x \in X \setminus F$. Then there exist open sets U and V of X such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$;
   (2) Let K be a compact subset of X and F a closed subset of X with $K \cap F = \emptyset$. Then there exist open sets U and V of X such that $K \subseteq U$, $F \subseteq V$, and $U \cap V = \emptyset$.
5. Let X be an LCS defined by a family of seminorms $\mathbb{P}$. Let $A \subseteq X$. Prove that the following are equivalent:
   (1) The subset A is bounded;
   (2) For any continuous seminorm q on X, $\sup\{q(a) : a \in A\} < \infty$;
   (3) For any $p \in \mathbb{P}$, $\sup\{p(a) : a \in A\} < \infty$.
6. Prove the following:
   (1) Assume that X is a finitely dimensional vector space and that (X, τ_1) and (X, τ_2) are two Hausdorff TVS, then $\tau_1 = \tau_2$;
   (2) A Hausdorff TVS is finitely dimensional if and only if it is locally compact.
7. Prove the following:
   (1) Any vector subspace of a TVS is also a TVS with respect to the subspace topology;
   (2) The closure of any subspace of a TVS is also a subspace;
   (3) Any finitely dimensional subspace of a Hausdorff TVS is closed.
8. Let X and Y be two LCS. Let $\mathbb{Q}$ be a family of seminorms that define the topology of Y. Let $T : X \to Y$ be a linear map. Prove that the following are equivalent:
   (1) $T : X \to Y$ is continuous;
   (2) For any continuous seminorm p on Y, $p \circ T$ is a continuous seminorm on X;
   (3) For any $q \in \mathbb{Q}$, $q \circ T$ is a continuous seminorm on X.
9. Let X be an LCS topologized by a sequence of seminorms $\{p_n\}_{n=1}^{\infty}$ on X such that $\cap_{n=1}^{\infty} \{x \in X : p_n(x) = 0\} = \{0\}$. Let
   $$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)} \qquad \forall x, y \in X.$$
   Prove that (X, d) is a metric space and its topology is the same as that induced by the seminorms $\{p_n\}_{n=1}^{\infty}$.
10. Let X be an LCS. Prove that X is metrizable if and only if X is first countable.