Math 241B: Functional Analysis
Winter, 2007

Homework Assignment 2
Due Wednesday, February 8

1. Let X be an LCS, M a closed subspace of X, and $x \in X \setminus M$. Prove that there exists $f \in X^*$ such that $f(M) = \{0\}$ and $f(x) = 1$.
2. Let X be a TVS over the field $\mathbb{F}$. Prove the following:
   (1) If $\mathbb{F} = \mathbb{R}$ and $f \in X^*$ with $f \neq 0$, then $X \setminus \ker f$ has two components $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$;
   (2) If $\mathbb{F} = \mathbb{C}$ and $f \in X^*$ with $f \neq 0$, then $X \setminus \ker f$ is path connected.
3. Let X be an LCS. Prove the following:
   (1) The topology $\sigma(X, X^*)$ is the smallest topology of X such that each $x^* \in X^*$ is continuous;
   (2) The topology $\sigma(X^*, X)$ is the smallest topology of X^* such that each $x \in X$ is continuous.
4. Let $f_k(x) = \sin kx$ $(0 \le x \le 2\pi)$ for $k = 0, 1, \ldots$. Prove the following:
   (1) The sequence $\{f_k\}_{k=1}^{\infty}$ converges weakly but not strongly to f_0 in $L^2[0, 2\pi]$;
   (2) The set $\{f_1, f_2, \ldots\}$ is strongly but not weakly closed in $L^2[0, 2\pi]$.
5. Let X be a normed vector space. Prove the following:
   (1) If $x_k \to x$ in X, then $x_k \rightharpoonup x$ and $\|x_k\| \to \|x\|$;
   (2) If $x_k \rightharpoonup x$ in X, then $\liminf_{k\to\infty} \|x_k\| \ge \|x\|$ and $\sup_{k\ge1} \|x_k\| < \infty$;
   (3) If $x_k \rightharpoonup x$ in X, then there exists $y_k \in \mathrm{co}\,\{x_i : 1 \le i \le k\}$ such that $y_k \to x$ in X;
   (4) If $x_k^* \to x^*$ in X^*, then $x_k^* \overset{*}{\rightharpoonup} x^*$ and $\|x_k^*\| \to \|x^*\|$;
   (5) If $x_k^* \overset{*}{\rightharpoonup} x^*$, then $\liminf_{k\to\infty} \|x_k^*\| \ge \|x^*\|$ and $\sup_{k\ge1} \|x_k^*\| < \infty$.
6. A uniformly convex normed vector space X is a normed vector space whose norm $\|\cdot\|$ satisfies the following condition: for any $\varepsilon \in (0, 2]$, there exists $\delta = \delta(\varepsilon) > 0$ such that the conditions $\|x\| = \|y\| = 1$ and $\|x - y\| \ge \varepsilon$ imply that $\|(x + y)/2\| \le 1 - \delta$.
   (1) Prove that a Hilbert space is a uniformly convex Banach space.
   (2) Assume $x_k \rightharpoonup x$ and $\|x_k\| \to \|x\|$ in a uniformly convex normed vector space X. Prove that $x_k \to x$ in X.
7. Prove the following:
   (1) Any weakly open neighborhood of 0 of an infinitely dimensional Hausdorff TVS X contains an infinitely dimensional subspace of X;
   (2) If X is an infinitely dimensional normed vector space, then the weak closure of $\{x \in X : \|x\| = 1\}$ is $\{x \in X : \|x\| \le 1\}$.
8. Let X be a Banach space, Y a normed vector space, and $\mathbb{B}(X, Y)$ the normed vector space consisting of all bounded linear operators from X to Y. Let $\mathbb{A} \subseteq \mathbb{B}(X, Y)$. Assume that $\{Tx : T \in \mathbb{A}\}$ is weakly bounded in Y. Prove that $\mathbb{A}$ is norm bounded in $\mathbb{B}(X, Y)$.
9. Let X be a reflexive Banach space and M a closed subspace of X. Prove that the subspace M and the quotient space X/M are also reflexive.
10. Let X be a reflexive Banach space and A a nonempty, closed, convex subset of A. Let $I : A \to \mathbb{R}$ be a functional. Assume the following:
    (1) There exist $\alpha > 0$ and $\beta \ge 0$ such that $I[u] \ge \alpha\|u\| - \beta$ for all $u \in A$;
    (2) The functional $I : A \to \mathbb{R}$ is sequentially weakly lower semicontinuous.
    Prove that there exists $u \in A$ such that $I[u] = \inf_{v\in A} I[v]$.